Ground State and Spin-Glass Phase of the Large-N Infinite-Range Spin Glass via Supersymmetry

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The large-N infinite-range spin glass is considered, in particular, the number of spin components k needed to form the ground state and the sample-to-sample fluctuations in the Lagrange multiplier field on each site. The physical significance of k for the correlation functions is discussed. The difference between the large-N and spherical spin glass is emphasized; a slight difference between the average Lagrange multiplier of the large-N and spherical spin glasses is derived, leading to a slight increase in the energy of the ground state compared to the naive expectation. Further, there is a change in the low-energy density of excitations in the large-N system. A form of level repulsion, similar to that found in random matrix theory, is found to exist in this system, surviving interactions. Even though the system is an interacting one, a supersymmetric formalism is developed to deal with the problem of averaging over disorder.

I. INTRODUCTION

Spin glasses have been extensively investigated for a long time.[1, 2] Spin glasses with finite N (the definition of N is discussed below) are dealt with using a replica technique which works wonderfully in the infinite range model. Unfortunately, it is known from other problems in random matrix theory[9] that replica techniques sometimes fail, especially for nonperturbative properties. In this paper we look at the large-N spin glass, and show that there exists a supersymmetric technique for dealing with this system. This system is a disordered, interacting (although interacting only in the sense of the large-N limit) system, that can still be dealt with in a fairly rigorous fashion.

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The large-N generalization of the spin glass that we use is not new, although we will show that the behavior is much more complicated than has been found using replica techniques on this problem.[3] For example, we define a number k, the number of spin components used to form the ground state, and find a non-trivial scaling of k with the system size; in the replica technique k is not even defined. Further, we will show that k is closely related to the correlation functions in the spin-glass phase, and that these correlation functions vary in a power law with the system size; in the previous replica work, the only method of obtaining correlation functions is from the mean-field Edwards–Anderson order parameter, which will give the wrong result. We explain some of the reasons for this error.

We will find significant differences between the large-N spin glass and finite N spin glass, much greater than the differences between large-N and finite N unfrustrated systems. Many of these differences stem from the fact that while an unfrustrated system will have the same ground state for large-N as for finite N, the frustrated large-N systems will use a large number, k, of the available spin components; the finite N systems have far fewer spin components available.

Of course, all physical glassy systems will have $N = 1, 2$, or 3, and will not be large-N systems. The different behavior of the large-N system in the glassy case makes the large-N approximation less useful for glassy systems than it is for ordered systems. However, there may exist some optimization problems, or problems from computer science or combinatorics, which will lie in the large-N limit, at least for finite V. Also, it may be possible some-day to extend the techniques in this paper to the finite dimensional case, and obtain results on this system which cannot be obtained in the finite N, finite dimensional problem.

The infinite range model mentioned above is one of the simplest models of a spin glass. It is a system of V spins, all interacting with each either via Gaussian distributed random interactions. In the general case, we may let each spin have up to N components, so that the Hamiltonian is given by

$$H = \sum_{i,j,\mu} \phi_i^\mu \phi_j^\mu H_{ij} \tag{1}$$

where i, j index the various sites and range from 1 to V, while μ indexes various components of the spin and ranges from 1 to N. We impose the constraint that

$$\sum_\mu (\phi_i^\mu)^2 = N \tag{2}$$

We will take H_{ij} to be a real symmetric matrix for most of the paper. In Section XV we will discuss situations in which H may have complex entries and ϕ may also be complex.

We will consider the problem in the large-N limit. Consider the statistical mechanics of the system, integrating over all spin configurations with a weight $e^{-\beta H}$. It is well-known, and will be reviewed below, that in the large-N limit the solution of a self-consistency equation will provide the properties of the system at any given inverse temperature β and for any given matrix H_{ij}. A similar self-consistency equation will provide the ground state properties of the system.

The problem that will be considered in this paper is the ground state and thermodynamic properties of the Hamiltonian given by Eqs. (1) and (2), in the large-N limit, when H_{ij} is drawn from an ensemble of real, symmetric, Gaussian distributed matrices. There exist several solutions of such problems in the literature,[3] using replica techniques. However, there are several problems with such solutions. Later in the paper we will discuss technical problems with that work. However, here, let us simply mention that two important properties of the ground state (and the spin-glass phase) are not addressed in that work. One property is the number of spin components used to form the ground state. Another is the strength of sample-to-sample fluctuations in the Lagrange multiplier field used to define the large-N limit (discussed in a later section).

Consider the number of components used to form the ground state. It is clear that a system with large-N has more freedom to choose a ground state than a system with a smaller N. There are more possible directions to point the spin components. To give a simple example, consider a system with three sites, and an anti-ferromagnetic interaction between the sites, so that $H_{ij} = 1$ for all i, j. If $N = 1$, the lowest energy state is to take one site with spin $+1$ and two sites with spin -1, or vice-versa. If $N = 2$, one can find a lower energy state. For example, one can set $\phi_1 = \sqrt{2}(1, 0)$, $\phi_2 = \sqrt{2}(-1/2, \sqrt{3}/2)$, and $\phi_3 = \sqrt{2}(-1/2, -\sqrt{3}/2)$. The factor of $\sqrt{2}$ in front of all the spins is simply to conform with the normalization chosen in Eq. (2); the important point is that the spins now use two different components. For the same system, with $N > 2$, one will still find that the spins only need to use two components. In fact, for this system, for any N, the different spins will always span a 2 dimensional vectors space.

Let us define the number of spins used to form a given state. We will call this number k. We may define this number for any state as the dimension of the vector space spanned by the V different vectors ϕ_i^μ, where here as $i = 1 \cdots V$ the spins ϕ_i^μ define V different vectors lying in the N dimensional space of vectors ϕ^μ. The vector space spanned by these vectors is a k-dimensional subspace of this N-dimensional space.

The ground state of a system will have Goldstone modes, due to the possibility of rotating all the spins together. Ignoring these modes, a system will generically have a non-degenerate ground state; that is, all ground states will be linked by symmetry. In that case, the number of spins needed to form the ground state will be the dimension k of the space spanned by the vectors ϕ_i^μ used to form the ground state. However, we can imagine systems with degenerate ground states. For example, a system of V spins, with no interaction between the spins, has a very degenerate ground state manifold. All states have the same energy, so all states are ground states. The value of k for these states can be anything between 1 and V. When we refer to the number of spins needed to form the ground state, we will in this case use the smallest k of all the states on the ground state manifold.

An important question is how big k is for the ground state in the large-N limit, where the system can take k as large as it wants. One property we will find, due to the additional freedom to place spins in more directions for large-N, is the uniqueness of the solution to the thermodynamic self-consistency equation, and the absence of metastable states. This may be interpreted as an absence of replica symmetry breaking.

The central result of the paper is the relation between V and the number of spin components k used to form the ground state. We will show that $k(k+1)/2 \leqslant V$ for any given H_{ij}. Then we will show that the infinite range system system, for large V, has k of order $V^{2/5}$. So, the bound is not saturated.

Throughout the paper, we will be taking the large-N limit first, and then considering systems with large, but finite V. By addressing the number of spin components used to given V, we will then be able to comment on the nature of the large-N and large V limits. In fact, these limits are different, and the interchange of limits is not justified. One can guess that if N is of order $V^{2/5}$, or bigger, then the system will be in a regime described by large-N, while if N is, smaller than $V^{2/5}$ the behavior will be completely different. It is likely that replica symmetry breaking will occur only if N is smaller than $V^{2/5}$.

From results on random matrices,[5] the idea of level repulsion is common. We will show in Section II that for a given k, there will be a matrix which has k eigenvalues all of order $1/\beta$, so that in the large β limit these eigenvalues collide. This might seem to contradict the naive expectations of level repulsion; we will show in Section V that a modified form of level repulsion still exists in the system.

Another important result obtained in this paper is the magnitude of the sample-to-sample and site-to-site fluctuations in the Lagrange multiplier field λ_i used to define the large-N limit. This field also has a physical significance, connected with the total energy of the bonds attached to a

given site. This will provide information on the sample-to-sample variations in total energy. Within the replica formulations used previously, there are problems with calculating these fluctuations; this and other problems with the replica formulation will be discussed later.

From a standpoint of formal technique, it is nice that we can use supersymmetric techniques. This is a glassy, interacting system, and it is useful to have such a system that can be treated without using replicas. Although the replica technique is very powerful, it sometimes has certain mathematical difficulties that a supersymmetric technique does not. For calculation of the spectra and correlators of random matrices, the Efetov supersymmetry techniques are superior.[9] The supersymmetric technique used here is more closely connected to the supersymmetry introduced by Parisi and Sourlas.[8] However, we will make some connections to the Efetov supersymmetry technique in Section XIII.

The paper is set up as follows: first, we introduce the problem. In Section II we review the solution of the large-*N* problem via self-consistency. For any given H, we can exactly solve the problem by solving a self-consistency equation. The problem is to obtain results after averaging over different H. In Sections III and IV we review theorems[4] showing that the solution of the self-consistency equation is unique and bounding the value of k.

After this introduction and review of old results, we discuss some of what should be expected from the solution of the large-*N* spin glass. We discuss level repulsion, paramagnetic and spin-glass phases, and correlation functions. Section V introduces the idea of level repulsion in the interacting system. Section VI discusses the spherical model and Section VII discusses the large-*N* model in the paramagnetic phase. In the paramagnetic phase, these two models are equivalent in the thermodynamic limit. In the spin-glass phase they are very different, and this difference is emphasized. Section VIII discusses the consequences of the quantity k for the correlation functions and the physical significance of k.

Then we proceed to the actual calculation for the large-*N* system. In Section IX the supersymmetric formalism is derived. In Section X a simple example is discussed to help introduce some of the manipulations used in Section XI when the supersymmetric formalism is used to obtain results on the ground state. In Section XII we discuss the physical meaning of these results from Section XI, and argue that the results of that section are also valid for the spin-glass phase.

Finally, we make some connections with Efetov supersymmetry and replica techniques, and discuss extensions of the problem. Section XIII shows some interesting connections between the supersymmetric formalism of this paper and the Efetov supersymmetry. Section XIV reviews problems

with the replica solutions of the problem. Section XV discusses extensions of the results in this paper to more general matrices H, and Section XVI is a conclusion. At the very end there are two appendices containing some of the mathematical manipulations needed for the calculations of Section XI.

Finally, we will include a note on notation. Throughout, we will refer to vectors ϕ_i^μ or v_i^μ. The index i will range from $1 \cdots V$ and will index different sites in the problem. The index μ will range either from $1 \cdots k$ or from $1 \cdots V$ and can be used to index different spin components on each site, or to index one of k or V different eigenvectors (these vectors lie in the V-dimensional space of vectors ϕ_i). Eigenvalues will be written as E_μ. Lagrange multiplier fields which are a function of the site will be written as λ_i, c_i, and so on.

II. LARGE-N AND SELF-CONSISTENCY

In this section, we will review the solution of the large-N problem via a self-consistency equation. First we will discuss the solution of the thermodynamics of the problem at finite β. Then we will discuss the self-consistency equation for the ground state.

For the thermodynamics of the problem, we wish to compute the following partition function

$$Z = \int_{\sum_\mu (\phi_i^\mu)^2 = \beta N} d\phi_i^\mu e^{-H} \tag{3}$$

where

$$H = \sum_{i,\, j,\, \mu} \phi_i^\mu \phi_j^\mu H_{ij} \tag{4}$$

Here we have rescaled the spins by β. It is well known that one can introduce a Lagrange multiplier field λ_i and solve this problem by saddle point methods.

The saddle point equation for λ_i is

$$G_{ii} = \beta \tag{5}$$

where we define the Green's function matrix G by

$$(H_{ij} + \lambda_i)^{-1} = G \tag{6}$$

In the large-N limit, after solving for λ_i, then the field ϕ can be treated as a free Gaussian field, with Hamiltonian

$$\beta(H_{ij} + \lambda_i) \tag{7}$$

When solving for λ_i, we must find a λ_i such that

$$H_{ij} + \lambda_i > 0 \tag{8}$$

That is, $H + \lambda_i$ must be a positive definite matrix.

We can relate λ to a physical quantity, the energy. The average energy of the system is

$$\frac{V}{\beta} - \sum_{i=1}^{V} \lambda_i \tag{9}$$

We can take a zero temperature ($\beta \to \infty$) limit of the self-consistency Eq. (5) to find an equation defining the ground state of the system. In this limit, some number of eigenvalues of $H + \lambda_i$ must go to zero.

In the zero temperature limit of the problem, we can also solve the problem by giving a configuration of spins, ϕ_i^μ which minimizes the energy. In this case, then for some λ_i

$$(H_{ij} + \lambda_i \delta_{ij}) \phi_j^\mu = 0 \tag{10}$$

for all μ. The spin configuration must be made up of zero eigenvectors of $H_{ij} + \lambda_i \delta_{ij}$. If the spins span a k dimensional space, then there must be at least k such zero eigenvectors.

To give a simple example in which $k \neq 1$, consider the system of three spins interacting anti-ferromagnetically with each other, as discussed in the introduction. We have $H_{ij} = 1$ for all $i \neq j$. If we take $\lambda_i = 1$ for all i, we find that $H_{ij} + \lambda_i$ has two zero eigenvectors. Here we have a system with $k = 2$.

For large, but finite β, the matrix $H + \lambda_i$ will have k eigenvalues which scale to zero as $1/\beta$. These are the eigenvalues which will vanish in the zero temperature limit. The other $V - k$ eigenvalues will tend to non-zero limits in the limit of large β. If these eigenvalues are denoted E_μ for $\mu = 1 \cdots k$, with eigenvectors v_i^μ, then an acceptable ground state is

$$\phi_i^\mu = \sqrt{\frac{N}{\beta E_\mu}} v_i^\mu \tag{11}$$

In the thermodynamic limit, it is possible for there to be a phase transition as the temperature is lowered. In this case, $H + \lambda_i$ will have k eigenvalues of order k/V. Physically speaking, these are states which is

macroscopically occupied. For a frustrated system, in which k eigenvalues become macroscopically occupied, there will be k eigenvalues of order k/V.

We expect that the properties of the system on the ordered side of the phase transition will be similar to the zero temperature properties. At zero temperature, k eigenvalues becomes equal to zero, vanishing as $1/\beta$. For finite, but large, β, we have k eigenvalues of order k/V. For large V, the eigenvalues are all close to zero. So, it is reasonable to expect that the properties of the spin-glass phase should be similar to the zero temperature problem.

In the simple example of a ferromagnet, we had a gap between the one eigenvalue which was macroscopically occupied and the $V-1$ eigenvalues above it. This will not remain true in the spin glass systems considered in this paper. There will be a spectrum of excitations above the k eigenvalues which are macroscopically occupied. This spectrum will have a small gap which vanishes as a power law in the large V limit; we will derive this gap in Section XI and Appendix A. All the states above the gap can be treated as a continuum and integrated over when obtaining the Green's function; there will be k states below the gap which must be treated more carefully.

III. UNIQUENESS OF SELF-CONSISTENT SOLUTION

Given the self-consistency equation defined in the previous section, we will show that the solution of this equation is unique. That is, we have the equation for all i

$$G_{ii} = \beta \tag{12}$$

where we define the Green's function matrix G by

$$(H_{ij} + \lambda_i)^{-1} = G \tag{13}$$

We will show that for any H there is only one solution λ_i of this equation which has the property that $H_{ij} + \lambda_i$ is a positive definite operator.

The proof of this has been given before,[4] so we will only review it here. The proof proceeds as follows: assume for some given H, we have a λ_i that solves the self-consistency equation, with $H + \lambda_i$ positive definite. One can show that for any infinitesimal change dH in H, there find a change $d\lambda_i$ in λ_i such that the self-consistency equation is still obeyed. Further, one can show that the change in λ_i is well-behaved; that is, that the differential equation found leads to $d\lambda_i/dH_{ij}$ always being finite. These two results imply that starting from any given H, we can deform H from the initial H to the point $H = 0$, and correspondingly deform λ_i along this

path, with the deformation of λ_i along the path being unique and well-behaved. However, for $H = 0$, the only solution is clearly $\lambda_i = 1$ for all i. Now, if the initial H were to have two different solutions to the self-consistency equation with different λ_i, then there would be a contradiction, since then we could find two different solutions to the self-consistency equation for $H = 0$ by deforming the two different solutions from the initial H to $H = 0$. So, the theorem will be proved.

IV. MAXIMAL NUMBER OF COMPONENTS TO FORM THE GROUND STATE

We will derive a bound on the maximal number of components needed to form the ground state of an arbitrary. Hamiltonian for a system of V spins. The system will be in the large-*N* limit so that it has as many components available as it needs. This theorem has been given before,[4] but it will be reviewed here. We will consider here only the case of real Hamiltonians; others will be discussed in the last section. The result for real Hamiltonians is that k, the number of components needed, satisfies the inequality $k(k+1)/2 \leqslant V$. At the end of this section we will explicitly construct a sequence of systems which saturate this bound for any k.

The proof of the inequality $k(k+1)/2 \leqslant V$ is very simple. Suppose the system needs k spin components to form the ground state. Then, for small perturbations δH of the Hamiltonian H, the system will continue to need k spin components to form the ground state. However, as shown above in the discussion of the large-*N* self-consistency equation, this means that the operator $H + \lambda_i$ has at least k zero eigenvalues. For a real, symmetric matrix, such as $H + \lambda_i$, to have k eigenvalues requires adjusting a total of $k(k+1)/2$ parameters. Since we have assumed that for any small perturbation of H the system continues to have k zero eigenvalues, the only parameters available to tune to maintain the k zero eigenvalues are the V different values of λ_i. Therefore, we have that $k(k+1)/2 \leqslant V$.

Note that this result is not a result for an ensemble of H; it is a result for any H. To express the proof more formally, let H be a V-by-V matrix. The dimension of the space of δH is $V(V+1)/2$. The space of matrices $(H + \delta H + \lambda_i + \delta \lambda_i)$ that satisfy Eq. (5) has dimension $V(V+1)/2 - V$ since we have V parameters $\delta \lambda_i$ to adjust to satisfy the V self-consistency equations. However, the space of V-by-V matrices with k zero eigenvalues has dimension $V(V+1)/2 - k(k+1)/2$. If we need to use k spin components to form the ground state, then for small δH every matrix $(H + \delta H + \lambda_i + \delta \lambda_i)$ has k zero eigenvalues. So a space of dimension $V(V+1)/2 - V$ is a subspace of a space of dimension $V(V+1)/2 - k(k+1)/2$; this is only possible if $k(k+1)/2 \leqslant V$.

For non-generic Hamiltonians H, we can have k zero eigenvalues of $H + \lambda_i$, with $k(k+1)/2 > V$. However, this requires also adjusting some parameters in the matrix H to produce these extra eigenvalues. Since we have assumed above that for arbitrary small perturbations of H there will still be k eigenvalues, we have excluded this non-generic case. The case in which $k(k+1)/2 > V$ is a case in which the Hamiltonian has a degenerate ground state, as discussed in the introduction. However, the number of spins needed to form the ground state will still satisfy $k(k+1)/2 \leqslant V$.

One immediate consequence of this that for sufficiently large-N, the system cannot have any metastable states. We have shown above that for any V, we can construct a ground state using k spin components where $k(k+1)/2 \leqslant V$. More generally, we have shown that any local minimum can be constructed using k spin components with $k(k+1)/2 \leqslant V$. Let us suppose we have an N component system, where N is equal to $2k_{\max}$ where $k_{\max}$ is the largest k which obeys $k(k+1)/2 \leqslant V$. Suppose we have a local minimum of the system, ϕ_i^μ, which is distinct from the ground state ρ_i^μ. Since we have $2k_{\max}$ total spin components, we can arrange things so that ϕ_i^μ is nonvanishing only for $\mu = 1 \cdots k_{\max}$, while ρ_i^μ is nonvanishing only for $\mu = k_{\max} + 1 \cdots 2k_{\max}$. Then, consider the state $\sqrt{1 - \delta^2}\, \phi + \delta\rho$. As δ is changed from 0 to 1, this provides a path starting from ϕ such that, near $\delta = 0$, the change in energy vanishes to first order in δ but is negative to second order in δ. So, ϕ_i could not have been a local minimum, and that for $N \geqslant 2k_{\max}$ there are no local minima other than the ground state.

Finally, we would like to show that the bound derived above for the maximum k as a function of V is sharp. That is, for any k, we will construct a system using $V = k(k+1)/2$ spins that employs k spin components to form the ground state. The procedure is discussed below and illustrated in Fig. 1.

First let us look at systems with small k. For $k = 1$, we have $k(k+1)/2 = 1$. Obviously, a system consisting of one spin always employs one spin component. To find systems with higher k, we will proceed
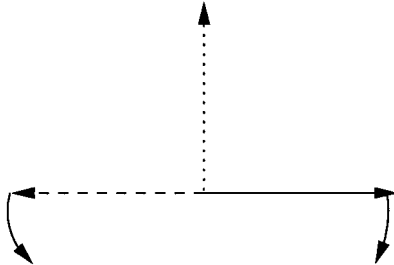

Fig. 1. Illustration of the construction of a series of systems saturating the bound.

inductively. Assume that, for some k, there is an H_{ij} acting on a system of $V_k = k(k+1)/2$ spins such that the ground state uses k spin components. We will construct a Hamiltonian acting on a system of $V_{k+1} = (k+1) \times (k+2)/2$ spins which uses $k+1$ spin components. Note that $V_{k+1} - V_k = k+1$. That is, we have an additional $k+1$ spins to use.

Let the ground state of the system of V_k spins be ρ_i^μ where $i = 1 \cdots V_k$. Assume, without loss of generality, that the vectors ρ_i for $i = 1 \cdots k$ span a k-dimensional vector space. Consider the following Hamiltonian

$$H = \sum_{i,\,j=1}^{V_k} \phi_i^\mu \phi_j^\mu H_{ij} + \sum_{i=1}^{k} \phi_i^\mu \phi_{V_k+i}^\mu + \varepsilon \sum_{i=1}^{k} (\phi_i^\mu + \phi_{V_k+i}^\mu)\, \phi_{V_{k+1}} \quad (14)$$

where ε is taken to be sufficiently small.

We have added $k+1$ spins to the previous Hamiltonian. Each of the first k of these spins (these are the spins with $i = V_k + 1 \cdots V_k + k$) has an anti-ferromagnetic interaction with one of the spins in the first V_k spins. Let us consider this system consisting only of the original V_k spins and these first k additional spins. This system has $V_k + k = V_{k+1} - 1$ spins. The ground state of this system is clear. The first V_k spins are in the ground state of the original system ($\phi_i^\mu = \rho_i^\mu$), and the next k spins each point opposite to one of the spins in the first V_k spins. This produces k pairs of spins, such that the spins in a pair point in opposite directions. For example, $\phi_1^\mu = -\phi_{V_k+1}^\mu$, $\phi_2^\mu = -\phi_{V_k+2}^\mu$, and so on.

Finally, we add the last spin, to produce a total of V_{k+1} spins. This one additional spin has a weak anti-ferromagnetic interaction with the $2k$ spins in these k pairs. Starting with the first $V_k + k$ spins fixed in their ground state, adding the V_{k+1}th spin does not change the energy at all because the anti-ferromagnetic interactions cancel. However, in a given pair of spins, one can slightly bend the pair, so that the two spins form an angle of less than π, and one can gain energy from adding the V_{k+1}th spin. Since ε is small, the energy gain is small. However, if the angle between two spins in a pair is $\pi - \delta$, than the energy gain is first order in δ, while the energy cost is second order in δ. So, even for small ε, it is advantageous to distort the pairs of spins. As chosen above, the first k spins span a k-dimensional space.

To distort a pair and gain energy, each spin in that pair has to have some component of its distortion in a direction opposite to the direction in which the V_{k+1}th spin lies. Also, for small distortions the direction in which a spin distorts must be orthogonal to the direction in which it points, since all spins must have fixed length. So, since each pair must distort in a direction orthogonal to that in which it lay orignally, and $\phi_{V_{k+1}}$ must have some component in the direction in which that pair distorts, we

find that the V_{k+1}th spin must have some component which is orthogonal to ϕ_i for $i = 1 \cdots k$. Then, the system of V_{k+1} spins must span a $k+1$-dimensional space, as needed.

In Fig. 1, we illustrate this procedure, in the simple case of the transition from $k = 1$ to $k = 2$, going from $V = 1$ to $V = 3$. A solid arrow is used to illustrate the spin in the original system (the $k = 1$, $V = 1$ system), a dashed arrow is used to illustrate the spin which aligns opposite to that spin, a dotted arrow is used to illustrate the V_{k+1}th spin (the third spin), and small arrows are used to illustrate the direction of distortion of the first two spins after the third spin is added.

V. LEVEL REPULSION

An interesting question is the existence of level repulsion for the large-N system. We know that if H_{ij} is dawn from an ensemble of real, symmetric matrices that its eigenvalues will repel each other.[5] This repulsion is for a noninteracting system. Stated technically, the level repulsion implies that two make two eigenvalue coincide requires adjusting two parameters, instead of just one as would be thought. What we will show in this section is that, for appropriately scaled eigenvalues, one must also adjust two parameters to make the eigenvalues coincide. We will show it precisely only for a three spin system, but at the end we will discuss generalizations to an arbitrary number of spins. Made precise, this is a statement about the dimension of the space of Hamiltonians with two coinciding eigenvalues.

For the interacting large-N system we are considering, the, eigenvalues of $H + \lambda_i$ will be different from those of H. The solution of the self-consistency equation can force k of those eigenvalues to scale to zero as $1/\beta$. This means that these eigenvalues become very close to each other, differing only by an amount of order $1/\beta$. Let us denote these eigenvalues by E_μ, where $\mu = 1 \cdots k$. Among those k eigenvalues,, there still is a kind of level repulsion. Consider a problem in the limit $\beta \to \infty$, so the solution to the problem is a configuration of spins ϕ_i^μ defining the ground state of the system. By a rotation of the spins, we can assume that

$$\sum_{i=1}^{V} \phi_i^\mu \phi_i^\nu = 0 \qquad (15)$$

Then, by permutation of the spin components μ we can arrange it that

$$\sum_{i=1}^{V} (\phi_i^\mu)^2 = \frac{N}{\beta E_\mu} \qquad (16)$$

As $\beta \to \infty$, $E_\mu \to 0$ for $\mu = 1 \cdots k$ but βE_μ stays non-zero.

Now, we will show that, for a three spin model, there exists a level repulsion between the numbers βE_μ in the large β limit, so that one must adjust two parameters in the Hamiltonian to make two of these coincide, instead of one parameters as one would think. Consider a system of three sins, interacting with some matrix H_{ij}. We know that if $H_{ij} = 1$ for all $i \neq j$ then we have the antiferromagnetic system discussed in the introduction. It can be checked that for this system

$$\beta E_1 = \beta E_2 \tag{17}$$

This system has $k = 2$ and both of the eigenvalues which go to zero are equal. So, we have two levels overlapping. We would like to know how many parameters are needed to guarantee that the two levels overlap. If only one parameter is needed, then there is no level repulsion. If two parameters are needed, then there is level repulsion. The parameters we can vary are the different elements of H_{ij}; there are two independent parameters, since H is symmetric and the diagonal elements of H are unimportant, and also any multiplication of all elements of H by a common, positive factor does not change the ground state.

Even if we do not rotate the spin components to make Eq. (15) true, a simple equation for E_1, E_2 can still be derived. It can be shown that

$$\frac{1}{E_1} - \frac{1}{E_2} = \sqrt{\left(\sum_i (\phi_i^1)^2 - \sum_i (\phi_i^2)^2\right)^2 + \left(\sum_i 2\phi_i^1 \phi_i^2\right)^2} \tag{18}$$

Consider matrices H near the uniform antiferromagnetic H considered above. There are two independent parameters defining the matrix H, if all we are interested in is ground state properties. There are also two independent parameters defining any set of states equivalent under rotations, since there are three spins and one overall rotation mode of the three spins. It is easy to show that small variations in parameters in H produces small changes in ϕ_2, ϕ_3, with no singularities in the Jacobian relating changes in the two independent parameters of H to the two parameters defining ϕ_2, ϕ_3. So, instead of asking how many parameters of H must be tuned to produce $E_1 = E_2$, we can ask, when considering level repulsion, how many of the two independent parameters defining ϕ_2, ϕ_3 must be tuned to produce $E_1 = E_2$. Looking at Eq. (18), we see that we must have $\sum_i (\phi_i^1)^2 - \sum_i (\phi_i^2)^2 = 0$ and also $\sum_i 2\phi_i^1 \phi_i^2 = 0$. This requires tuning both available parameters to satisfy the two requirements.

So, we have shown level repulsion, in the three spin system, in the sense that two parameters must be adjusted to make levels coincide. One can consider a system with an arbitrary number of spins. In this case, one

can still show that one must adjust two parameters in the ground state distribution of spins to make two levels coincide. However, the dimension of the space of spin configurations will be less than that of H. So, the Jacobian connecting H to the space of ground states must have nonvanishing nullity (dimension of the nullspace). If one could show that the nullity were constant, regardless of the number of coinciding levels, than one could obtain a similar result which would show that the dimension of the space of H with two coinciding levels is 2 less than the dimension of the space of H with no coinciding levels. This result seems reasonable, but we have not fully shown it yet.

VI. SPHERICAL MODEL

A simplification of the large N spin glass is the spherical model. This will be a useful simple example. Here the system will always condense into one eigenvalue ($k = 1$). The spherical model is defined by using the Hamiltonian

$$H = \sum_{i, j} \phi_i \phi_j H_{ij} \tag{19}$$

subject to the constraint that

$$\sum_i (\phi_i)^2 = \beta V \tag{20}$$

Here, there is only one component of ϕ on each site, but the constraint is relaxed to the requirement that the sum of spins over all sites be equal to V. The problem is again solved by self-consistency, with the self-consistency equation $\mathrm{Tr}(G) = \beta V$ where $G = (H + \lambda)^{-1}$ where λ is now independent of the site index i.

Consider an ensemble of matrices H_{ij} given by the measure $dH \, e^{-V\tau \, \mathrm{Tr} \, H^2}$, where τ is some number of order unity. Averaged over different H_{ij}, we have[5]

$$G_{ii} = 2\tau\lambda - 2\sqrt{\tau^2\lambda^2 - \tau} \tag{21}$$

Also, we have that, on average,

$$\mathrm{Tr}(G) = 2V\tau\lambda - 2V\sqrt{\tau^2\lambda^2 - \tau} \tag{22}$$

Using this in equation, we find that

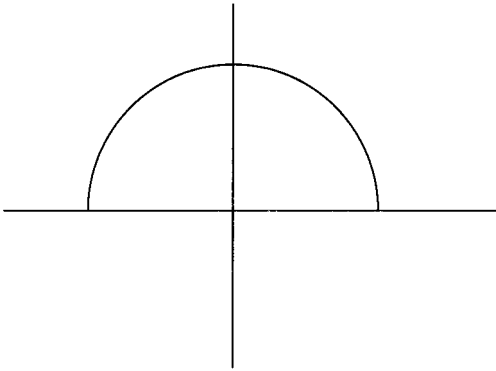$$\lambda = \frac{1}{\beta} + \frac{\beta}{4\tau} \tag{23}$$

Fig. 2.   Wigner semicircle distribution of eigenvalues. The horizontal axis is energy, the vertical axis is eigenvalue density. The semicircle extends from $-\sqrt{1/\tau}$ to $\sqrt{1/\tau}$.

Equation (22) can be obtained from the Wigner semicircle[5] distribution of eigenvalues of H, shown in Fig. 2. The eigenvalues, in the large V limit, range from $-\sqrt{1/\tau}$ to $\sqrt{1/\tau}$. The density of eigenvalues is given by $V\rho(E)$, where $\rho(E) = (4/\pi\tau)\sqrt{1/\tau - E^2}$.

At $\beta = 2\sqrt{\tau}$, we have $\lambda = \sqrt{1/\tau}$ from Eq. (23). At this point the spectrum of $H + \lambda$ becomes gapless, and a phase transition occurs. The sum over eigenvalues needed to define Tr G must now separately include the contribution from the one lowest eigenvalue of H. So, we have that

$$\text{Tr } G = \frac{1}{\lambda + E_{\min}} + 2V\tau\lambda - 2V\sqrt{\tau^2\lambda^2 - \tau} \tag{24}$$

where $E_{\min}$ is the lowest eigenvalue of H. The self-consistency Eq. (23) can be satisfied in the spin glass phase only if $\lambda + E_{\min}$ is of order $1/V$. In this phase, we will have $\lambda \approx 1/\tau$, so that $H + \lambda$ will be approximately gapless. More precisely, it is known[7, 5] that the lowest eigenvalue of the system, $E_{\min}$ will lie within a tail that extends to a distance of order $V^{-1/6}$ below $1/\tau$. Then, $\lambda = \sqrt{\tau}$ plus a correction of order $V^{-1/6}$.

VII. PHASES OF THE LARGE-*N* INFINITE RANGE MODEL

The infinite range, large N, model of a spin glass exhibits a transition at a finite β from paramagnet to spin glass. In the paramagnetic phase, the operator $H + \lambda_i$ has a gap. In this phase, the behavior is very similar to that of the spherical model considered above. In the spin glass phase, $H + \lambda_i$ becomes gapless in the thermodynamic (large V) limit. Assuming k

components are used to form the ground state of the system, so that the system condenses into k states, their the operator $H + \lambda_i$ will have k eigenvalues of order k/V, as well as a spectrum of higher eigenvalues. First, will briefly discuss the infinite range model in the paramagnetic phase.

Suppose, as an approximation, that we can look at a system in which λ_i is the same for all values of i. This is essentially the spherical model considered above. Let this value of λ_i be denoted λ. Then, we find that

$$G_{ii} = ((H + \lambda)^{-1})_{ii} \tag{25}$$

Using Eq. (21), and requiring that $G_{ii} = \beta$, we find the same solution of the self-consistency equation as was found in Eq. (23). Further, it may be shown that the mean-square fluctuations in G_{ii} between different H_{ij}, for this constant λ are of order $1/V$.

However, in the paramagnetic phase, it is not true that λ_i is a constant. The slight fluctuations in G_{ii} away from the average value of Eq. (21) require λ_i to vary from site to site and sample to sample. They produce the difference between the large N model and the spherical model in this phase.

We can estimate the fluctuations in λ_i. The mean-square fluctuations in G_{ii} from site to site and sample to sample are of order $1/V$. So, the approximation of taking a constant λ will almost satisfy the self-consistency equation, but will be slightly in error due to these fluctuations in G_{ii} away from the average value. We will fix this by making a small change in λ_i. Let us consider the change in G_{ii} resulting from a change in λ_j. This is equal to

$$\delta G_{ii} = G_{ij} \delta \lambda_j G_{ji} \tag{26}$$

For $j \neq i$, this vanishes in the thermodynamic limit since $G_{ij} G_{ji}$ is of order $1/V$ (this will be discussed in the next section). For $j = i$, this yields

$$\delta G_{ii} = \delta \lambda_i \beta^2 \tag{27}$$

So, if a constant λ produces a small error in the self-consistency equation, we can fix it with a small change in λ_i on each site. We use Eq. (27) to estimate the change in λ_i needed. Since the mean-square fluctuations in G_{ii} are of order $1/V$, the mean-square fluctuations in λ_i must also be of order $1/V$. The fact that fluctuations in λ_i are small, and that fluctuations in G_{ii} are small for fixed λ, is what permits us to approximate the large N results by the spherical results in this phase.

In the spin glass phase, the spherical model and the large N model differ greatly. In the spherical model, the system condenses only into one

eigenvalue. The eigenvector for this eigenvalue is a randomly chosen vector from a V-dimensional space of vectors. So, G_{ii} can be written as the sum of two parts. The first is a contribution from the continuum of eigenvalues above the lowest eigenvalue; this part does not fluctuate strongly from sample-to-sample (when the spectrum of $H + \lambda$ becomes gapless, this part acquires mean-square fluctuations which are much bigger than in the paramagnetic phase, but which are still small in the large V limit). The second is a contribution from the lowest eigenvalue. This part provides a contribution to G_{ii} of order $V(v_i)^2$, where v_i is the eigenvector corresponding to this eigenvalue. This contribution to G_{ii} has fluctuations from sample-to-sample of order unity, as $(v_i)^2$ has site-to-site and sample-to-sample fluctuations of order $1/V$. So, the spin glass phase and ground state of the large N model will differ greatly from the spherical model. The nature of the ground state is the problem that will be addressed with supersymmetry techniques later in the paper. It may be assumed that the state in the spin glass phase for $\beta \geqslant 2\sqrt{\tau}$ is very similar to the ground state.

Using Eqs. (9) and (23), we find that the energy of the system in the paramagnetic phase is given by

$$- V \frac{\beta}{4\tau} \tag{28}$$

At the phase transition point, this becomes $-V(1/2\sqrt{\tau})$. In the spin glass phase, $\lambda \approx \sqrt{1/\tau}$, so that the energy is given by

$$V \frac{1}{\beta} - V \sqrt{\frac{1}{\tau}} \tag{29}$$

The number of spin components k is in a sense an analogue of Griffiths effects, known from finite dimensional systems, in the infinite range model. Consider a finite dimensional system of Bose particles with repulsion. As the temperature is lowered, there can be a phase transition to a superfluid phase. If there is disorder, there will also be a Griffiths phase, when the system has gapless excitations, but still has no long-range order. We can understand the Griffiths phase as follows: some single particle eigenstate of the finite dimensional system will be lowest in energy. The particles will begin to condense into this eigenstate. However, the eigenstate is localized, and finite in size, so only a microscopic number of particles can condense into the state before the interparticle repulsion raises the energy of this state, and particles begin to condense into some other state. This leads to a large number of states near zero energy, as the states

lowest in energy have their energy raised by interactions, and produces the gapless spectrum. In the infinite-range spin glass we consider in this paper, the Griffiths phase is not a separate phase, but the system still manages to produce a large number of very low energy states (this number is k) by adjusting λ_i.

In the large-N case, we showed that in the spin glass phase the approximation of using constant λ_i fails badly. The system system must shift λ_i to satisfy the self-consistency equation. In the process of shifting, the system will raise the energy of the lowest eigenstate of $H + \lambda_i$ more than it raises the energy of the eigenstates just above it. This means that several of the eigenvalues of $H + \lambda_i$ can become of order $1/V$, which leads to $k > 1$. This process is very similar to the formation of Griffiths states discussed above.

VIII. CORRELATION FUNCTIONS

Let us suppose we have a system with a non-degenerate ground state which uses k spin components. We would like to investigate the meaning of k and how one can see the effects of a given k by looking at the correlation functions of the system. Of course, we have defined k as the dimension of the vector space spanned by the vectors ϕ_i^μ. This is some subspace of an N-dimensional vector space. So, if one looks at all V vectors ϕ_i, one can determine what k is. Let us instead look at correlation functions of the system.

First, let us consider the correlation functions within the paramagnetic phase discussed in the previous section. Within the large-N formalism, the correlation function

$$\sum_\mu \langle \phi_i^\mu \phi_j^\mu \rangle \tag{30}$$

is equal to

$$NG_{ij} \tag{31}$$

where the average in Eq. (30) is a thermodynamic average for a given Hamiltonian. By self-consistency, we have that $\sum_\mu \langle \phi_i^\mu \phi_i^\mu \rangle = 1$. Now consider G_{ij} for $i \neq j$, in the paramagnetic phase. Clearly, this Green's function vanishes after averaging over different Hamiltonians. However, we can compute the mean-square fluctuations in this Green's function, where the fluctuations are from sample-to-sample.

The operator $H + \lambda_i$ has V different eigenvectors. Denote these eigenvectors by v_i^μ, where $\mu = 1 \cdots V$ now labels the different eigenvectors. Let each eigenvector have eigenvalue E_μ. Then

$$G_{ij} = \sum_\mu \frac{1}{E_\mu} v_i^\mu v_j^\mu \qquad (32)$$

Averaging over different matrices H_{ij} will cause the eigenfunctions to have random sign on each site, and will cause this quantity to vanish. Further,

$$G_{ij} G_{ji} = \sum_{\mu, \nu} \frac{1}{E_\mu E_\nu} v_i^\mu v_i^\nu v_j^\mu v_j^\nu \qquad (33)$$

Averaging over different matrices H_{ij}, we find that the right-hand side of Eq. (33) will vanish, except for the terms in which $\mu = \nu$. Then we obtain

$$G_{ij} G_{ji} = \sum_\mu \frac{1}{(E_\mu)^2} (v_i^\mu v_j^\mu)^2 \qquad (34)$$

Since we normalize v_i^μ by taking $\sum_i (v_i^\mu)^2 = 1$, we find that $(v_i^\mu)^2 \approx 1/V$. So, the right-hand side of Eq. (33) contains V terms of order $1/V^2$, and so $G_{ij} G_{ji}$ is of order $1/V$.

Now, consider the spin glass phase of the system. There will be k eigenvalues which are of order k/V. We can again write

$$G_{ij} G_{ji} = \sum_\mu \frac{1}{(E_\mu)^2} (\phi_i^\mu \phi_j^\mu)^2 \qquad (35)$$

as before. However, we can separate out from this sum the terms involving the k eigenvalues E_μ of order k/V. Then, we find that the right-hand side of Eq. (35) contains k terms of order $1/k^2$. So, $G_{ij} G_{ji}$ is of order $1/k$.

IX. SUPERSYMMETRIC FORMALISM

Here we set up a supersymmetric formalism to determine properties of a large-N system, averaged over disorder. In the Section XI, this formalism will be used to obtain results on the value of k for the ground state, the strength of fluctuations in λ_i, and other results. In this section, we will first derive the formalism in generality for any large-N system, and then discuss simplifications in the low temperature limit.

To average over quenched disorder, many tricks have been invented in other problems. The replica trick and Efetov supersymmetry both attempt to ensure that the partition function of a system, for a fixed realization of

disorder, is equal to unity. Then, one can average over different realizations of the disorder directly. Our goal will be similar. We wish to find a method of obtaining λ_i as a function of H. We will express this using an integral over all λ_i, such that the integral is equal to unity for any H.

First we will discuss a naive attempt which does not quite work. Then we will introduce a supersymmetry and give the full formalism.

Consider a given H_{ij}. The goal is to find a set of λ_i which satisfies the self-consistency equation $G_{ii} = \beta$. One simple way of implementing this might be to integrate over all λ_i with a set of δ-function constraints on the Green's function. We will also replace Eq. (8) with a positivity constraint on G instead. Writing these δ-function constraints as a set of integrals over variables c_i, we obtain

$$\int_{G>0} d\lambda_i \frac{dc_i}{2\pi} e^{ic_i(G_{ii}-\beta)} \tag{36}$$

where $G_{ii} = (H_{ij} + \lambda_i \delta_{ij})_{ii}^{-1}$. However, the above equation is not quite right as there is a non-trivial Jacobian coming from the δ-function. To find this Jacobian, assume we have found the set of λ_i which satisfy the self-consistency equation. Considering a shift $\delta\lambda_i$, we would find $\delta G_{ii} = G_{ij} \delta\lambda_j G_{ji}$.

Therefore, the integral over λ_i and c in Eq. (36) is equal to $1/\det(M_{ij})$, where M_{ij} is the matrix given by

$$M_{ij} = G_{ij} G_{ji} \tag{37}$$

Note that this is not a matrix square, but a square of individual elements in G_{ij} to produce M_{ij}. To cancel this determinant, we can add an additional set of integrals over Grassman $\bar{a}_i$, a_i variables to Eq. (36). This leads to a new equation

$$\int_{G>0} d\lambda_i \, dc_i \, d\bar{a}_i \, da_i \, e^{ic_i(G_{ii}-\beta)} e^{\bar{a}_i G_{ij} a_j G_{ji}} \tag{38}$$

The integral in Eq. (38) is equal to unity for any Hamiltonian H. Then, it is possible to average over Hamiltonians within the equation. The fact that the integral is equal to unity is due to a supersymmetry. There are $2V$ bosonic variables and $2V$ fermionic variables. Similar supersymmetries have been found useful in other systems.[8]

It is interesting to compare Eq. (38) to Eq. (26). The equations are similar; as discussed in reference to Eq. (26), in the paramagnetic phase the matrix M can be treated as diagonal in the thermodynamic limit. In the spin glass phase the matrix M will be very important and will be discussed below and in Section XI.

In the infinite range model, we can further simplify Eq. (38). Let us average over Hamiltonians H with Gaussian weight $e^{-V\tau \text{Tr } H^2}$. We obtain

$$\frac{\int_{G>0} d\lambda_i \, dc_i \, d\bar{a}_i \, da_i \, dH \, e^{-V\tau \text{Tr}(H^2)} e^{ic_i(G_{ii}-\beta)} e^{\bar{a}_i G_{ij} a_j G_{ji}}}{\int dH \, e^{-V\tau \text{Tr } H^2}} \tag{39}$$

For notational simplicity, we will not always write out the factor of $1/\int dH \, e^{-V\tau \text{Tr } H^2}$. This factor will be assumed throughout. At one point in Section X, and one point in Section XI, we will need this factor, but elsewhere it will be left out.

For given λ_i, we can write the integral over H as an integral over O and E with $H + \lambda_i = O^T E O$ (note that the left-hand side is $H + \lambda_i$, not H). Then $G = O^T E^{-1} O$. Then the integral of Eq. (39) can be written

$$\int_{E \geqslant 0} d\lambda_i \, dc_i \, d\bar{a}_i \, da_i \, dE \, dO \, e^{-V\tau \text{Tr}(H^2)} \prod_{\mu < \nu} |E_\mu - E_\nu| e^{ic_i(G_{ii}-\beta)} e^{\bar{a}_i G_{ij} a_j G_{ji}} \tag{40}$$

For given k, we will have k of the eigenvalues E_μ scaling as $1/\beta$. In the large β limit, these eigenvalues are much smaller than all other eigenvalues. It is useful to keep β finite, but in the large β limit we can also assume that G_{ii} is made up only of terms due to the k eigenvalues which go to zero, and ignore contributions to G_{ii} from other eigenvalues. Then, Eq. (40) simplifies. Let us define v_i^μ as the eigenvector corresponding to eigenvalue E_μ. In the large β limit, G_{ii} does not depend on v_i^μ for $\mu > k$. Further, for $\mu = 1 \cdots k$ and $\nu = k + 1 \cdots v$, we have $|E_\mu - E_\nu| = E_\nu$. Also, we can perform the integral over $\bar{a}_i, a_i$ to obtain $\det(M_{ij})$, where M_{ij} is defined by Eq. (37).

For given k, we will define a quantity Z_k. This is what is obtained if Eq. (40) is evaluated in the sector with k eigenvalues tending to zero. Equivalently, Z_k is the probability that, if we choose H from the ensemble at random, we will have k eigenvalues tending to zero. It must be the case that $\sum_k Z_k = 1$ as a result of the supersymmetry.

So, in the end we have

$$Z_k = \int dv_i^\mu \, dE_\mu \, \delta \left(\sum_i (v_i^\mu)^2 - 1 \right) \delta \left(\sum_i (v_i^\mu v_i^\nu) - 1 \right) Z_k^< [v_i^\mu, E_\mu] \, Z_k^> [v_i^\mu, E_\mu] \tag{41}$$

where

$$Z_k^< [v_i^\mu, E_\mu] = \int dc_i \, d\bar{a}_i \, da_i \prod_{\mu < \nu} |E_\mu - E_\nu| e^{ic_i(G_{ii}-\beta)} \tag{42}$$

and

$$Z_k^> [v_i^\mu, E_\mu] = \int dH_> \, d\lambda_i \, e^{-V\tau \, \mathrm{Tr}(H^2)} (\det(H_> + \lambda_i)^k \, e^{\bar{a}_i G_{ij} a_j G_{ji}} \tag{43}$$

This needs some explanation. The vectors v_i^μ are the eigenvectors associated with the eigenvalues E_μ for $\mu = 1 \cdots k$. The eigenvalues E_μ are all of order $1/\beta$. The Green's function is given by $G_{ii} = \sum_{\mu=1}^{k} (v_i^\mu)^2 (E_\mu)^{-1}$. The matrix $H_>$ is a V-by-V matrix, such that $H_> + \lambda_i$ has only $V - k$ non-zero eigenvalues. The zero eigenvectors of $H_> + \lambda_i$ lie in the subspace spanned by the v_i^μ. We have $H = H_> + \sum_\mu v_i^\mu E_\mu v_j^\mu$. That is, $H_> + \lambda$ is what results if one looks only at the $V - k$ largest eigenvalues of $H + \lambda$.

Equation (41) has been written using two equations: Eq. (42) and Eq. (43). Indeed, Eq. (41) integral almost separates into two parts. For convenience, we will refer to E_μ and v_i^μ, for $\mu = 1 \cdots k$ as the lower sector, while we will refer to the part of the integral involving $H_>$ as the upper sector. Similarly, we may refer to the upper and lower sectors of the matrix $H + \lambda_i$. One way in which the lower and upper sectors communicate is that the eigenvectors of the lower sector determine the allowed eigenvectors of the upper sector, which, if λ_i is not constant, can change the value of $Z_k^>$. Also, the determinant due to the integral over Grassman variables depends on both the upper and lower sector of eigenvalues.

It is interesting to see how the bound $k(k+1)/2 \leqslant V$ is realized in this formalism. We must show that for $k(k+1)/2 \geqslant V$ the integral of Eq. (41) vanishes in the large β limit. There are four pieces to this. First, the δ-functions, $\delta(\sum_{\mu=1}^{k} (v_i^\mu)^2 (E_\mu)^{-1} - \beta)$ can be rewritten as $(1/\beta) \, \delta(\sum_{\mu=1}^{k} (v_i^\mu)^2 \times (\beta E_\mu)^{-1} - 1)$. This then contributes a factor of $(1/\beta)^V$, as there are V such δ-functions. Remember that E_μ scales as $1/\beta$, so βE_μ tends to a constant in the large β limit.

Next, the level repulsion between the first k eigenvalues contributes a factor of $(1/\beta)^{k(k-1)/2}$. The integral over the first k values of E_μ contributes a factor of $(1/\beta)^k$, as E_μ is of order $1/\beta$. These two combine to produce $(1/\beta)^{k(k+1)/2}$.

Finally, there is the determinant due to the integral over the Grassman variables. This is the determinant of the matrix M_{ij}. First consider the part of this matrix due to the contribution of the first k eigenvalues to G. This is the determinant of

$$\sum_{\mu, \, \nu = 1}^{k} v_i^\mu v_i^\nu (E_\mu E_\nu)^{-1} \, v_j^\mu v_j^\nu \tag{44}$$

This is the sum of $k(k+1)/2$ distinct matrices, each matrix corresponding to a term in the above sum with given μ, ν. It can be seen that each of these

matrices has one nonvanishing eigenvalue which scales as β^2 in the large β limit. There is an additional contribution to the determinant which is equal to

$$2 \sum_{\mu=1}^{k} \sum_{\nu=k+1}^{V} v_i^{\mu} v_i^{\nu} (E_{\mu} E_{\nu})^{-1} v_j^{\mu} v_j^{\nu} \tag{45}$$

This is the sum of $k(V-k)$ distinct matrices each with one nonvanishing eigenvalue of order β. There is also a contribution in which both μ and ν are greater than k, but this is unimportant.

The determinant of the matrix M_{ij} is the product of the V eigenvalues of that matrix. There are at most $k(k+1)/2$ eigenvalues which are of order β^2. These arise from Eq. (44). The remaining $V - k(k+1)/2$ eigenvalues are of order β. So the determinant scales as $\beta^{V + k(k+1)/2}$.

Combining the various factors of $(1/\beta)^V$, $(1/\beta)^{k(k+1)/2}$, and $\beta^{V+k(k+1)/2}$, we find that everything cancels. However, if $k(k+1)/2 > V$, there will be a problem. It will not be possible for M_{ij} to have $k(k+1)/2$ eigenvalues of order β^2 since G has at most V eigenvalues. Therefore, if $k(k+1)/2 > V$, Eq. (41) will vanish in the large β limit.

X. A NON-INTERACTING EXAMPLE

In the previous section, the integral over $H + \lambda_i$ was written as an integral over a lower sector with eigenvalues of order $1/\beta$, and an upper sector. It will be useful to illustrate this technique of dividing an integral over random matrices into two sectors, using a simple example.

Let us consider an integral over all V-by-V matrices H, with weight $e^{-V\tau \operatorname{Tr} H^2}$. This integral is a Gaussian integral, and can be performed to yield

$$\int dH \, e^{-V\tau \operatorname{Tr} H^2} = \left(\frac{\pi}{V\tau}\right)^{-V/2} \left(\frac{\pi}{2V\tau}\right)^{V(V-1)/2} \tag{46}$$

We will separate out k eigenvalues from this matrix, breaking the integral into two sectors of eigenvalues. Then we will investigate the consequences of requiring that the integral over both sectors, including interaction between the sectors, is equal to Eq. (46).

Let us pick out k eigenvalues from H, and require them to be equal to E_{μ}, $\mu = 1 \cdots k$. Then, the integral over all matrices H can be written as

$$\int dv_i^\mu \, dE_1^\mu \, dH_> \prod_\mu \left[\det(H_> - E_\mu) \, e^{-V\tau E_\mu^2} \, \delta \left(\sum_i (v_i^\mu)^2 - 1 \right) \right]$$

$$\times \prod_{\mu < \nu} \left[|E_\mu - E_\nu| \, \delta \left(\sum_i v_i^\mu v_j^\nu \right) \right] e^{-V\tau \, \mathrm{Tr} \, H_>^2} \tag{47}$$

where $H_>$ is a $(V-k)$-by-$(V-k)$ matrix.

If $k = 1$, the eigenvalue distribution of $H_>$ will be almost unchanged from a Wigner semi-circle. Then, the term $(\det(H_> - E_\mu))^k \, e^{-V\tau E_\mu^2}$ is approximately equal to

$$\left(\frac{1}{2\sqrt{\tau}} \right)^V e^{-V/2} \tag{48}$$

for all E_μ inside the semicircle, $-1/\sqrt{\tau} \leqslant E_\mu \leqslant 1/\sqrt{\tau}$. We find

$$\int dH \, e^{-V\tau \, \mathrm{Tr} \, H^2}$$

$$\approx \int dE \left(\frac{1}{2\sqrt{\tau}} \right)^V e^{-V/2} \int dv_i \, \delta \left(\sum_i (v_i)^2 - 1 \right) \int dH_> \, e^{-V\tau \, \mathrm{Tr} \, H_>^2} \tag{49}$$

Therefore

$$\frac{\int dH \, e^{-V\tau \, \mathrm{Tr} \, H^2}}{\int dH_> \, e^{-V\tau \, \mathrm{Tr} \, H_>^2}} = \int dE \left(\frac{1}{2\sqrt{\tau}} \right)^V e^{-V/2} \int dv_i \, \delta \left(\sum_i (v_i)^2 - 1 \right) \tag{50}$$

The first ratio is equal to

$$\left(\sqrt{\frac{V\tau}{\pi}} \sqrt{\frac{2V\tau}{\pi}} \right)^{V-1} \tag{51}$$

The integral $\int dv_i \, \delta(\sum_i (v_i)^2 - 1)$ is approximately equal to

$$e^{V/2} \left(\frac{2\pi}{V} \right)^{V/2} \tag{52}$$

It can be verified that, if the last two equations are used in Eq. (50), that the factors of e^V and V^V cancel, as they should. A similar cancellation will be important in the next section.

If $k > 1$, the term $\prod_{\mu < \nu} |E_\mu - E_\nu|$ might seem to make Eq. (47) greatest when the eigenvalues E_μ are well separated. However, we know that the original matrix H has V eigenvalues, with level density of order V

and separation between levels of order $1/V$. For well separated levels, with $|E_\mu - E_\nu| \gg 1/V$, the effect of level repulsion is negligible. The solution to this problem lies in the determinants $\prod_\mu \det(H_> - E_\mu)$. For a given value of E_μ the determinant, $\det(H_> - E_\mu)$, will slightly alter the level distribution of $H_>$. This will produce a slight decrease in level density around E_μ. This will then, as a result of the determinant, $\det(H_> - E_\nu)$, produce an effective attraction between levels that cancels out $\prod_{\mu < \nu} |E_\mu - E_\nu|$ if $|E_\mu - E_\nu|$ is much greater than $1/V$.

Although this discussion seems elementary, it is worth reviewing this for what will happen in the next section. Then, the lower sector of $H + \lambda_i$ will have k eigenvalues all within order $1/\beta$ of each other. This will produce a change in the level density of $H_>$, and will change the determinant $(\det(H_> + \lambda_i))^k$ away from the naive value one would obtain by assuming that $H_>$ has a Wigner semicircle density of eigenvalues. The true level density of $H_>$ will be evaluated in the next section and in the appendices. Since k will be much greater than one, this change in level density will be very important.

XI. GROUND STATE FOR THE LARGE-*N* INFINITE RANGE MODEL

Having set up the supersymmetric formalism in the previous section, we will use this formalism to determine the most likely value of k and the strength of fluctuations in λ_i. It will be shown that the system will obey $k \propto V^{2/5}$. Further, it will be shown that the mean square fluctuations in λ_i are equal to $1/k\tau$. We will also derive a slight change in the average value of λ from $\sqrt{1/\tau}$, and a slight gap in the low energy excitations.

The starting point for the calculation is Eq. (41). We will evaluate this integral for different k and use this to determine the most likely k. For any given k, we can write the value of this integral as $Z_k = e^{S_k}$. Below we will evaluate various contributions to S_k. Due to the complexity of the calculations, we will only evaluate terms in S_k to order kV, $V \log V$, $V \log k$, V, $k^2 \log V$, and $k^2 \log k$. There will also be terms in S_k of order k^2, $V/\sqrt{k}$, and smaller, which we will not evaluate. All the terms of order kV, $V \log k$, $V \log V$, and V will cancel. It will turn out that the other terms do not cancel; given these terms, we will show at the end of the section that k is of order $V^{2/5}$, with the exact ratio $k/V^{2/5}$ undetermined without a computation of terms of order k^2 and $V/\sqrt{k}$ and in S_k. Note that for k of order $V^{2/5}$, then k^2 and $V/\sqrt{k}$ are of the same order.

The procedure to evaluate Eq. (41) will be to start with all λ_i equal to each other. We will then write $\lambda_i = \lambda$. In this case, we will evaluate the integral over the various eigenvalues E and eigenvectors v to obtain the

result. Then we will consider the effect of site-to-site fluctuations in λ_i at the end. From this we will get an effective action for λ which will tell us the average value of λ_i and the strength of fluctuations about this average.

The rest of this section will be divided into four subsections. First there will be a calculation of the contribution to Z_k from the upper sector. Then there will be a calculation of the contribution to Z_k from the lower sector. These two calculations will be for the case with all λ_i equal to a constant λ. In the third subsection, we will consider the effect of permitting λ_i to vary from site to site. In the final subsection, we will put the calculations of the first three subsections together and obtain final results.

A. Upper Sector

For constant $\lambda_i = \lambda$, Z_k is completely independent of the eigenvectors v_i^μ for $\mu = k+1 \cdots V$. Z_k does still depend on E_μ for $\mu = k+1 \cdots V$. The matrix $H + \lambda$ has k eigenvectors near zero and $V - k$ other eigenvectors which are all positive. The average value of λ is determined by a balance of two effects. First, there is a term in the probability distribution for λ (when all λ_i are set equal to λ) like $e^{-kV\tau\lambda^2}$. This term simply arises from the requirement that H must have k eigenvalues equal to $-\lambda$ and from the term in Eq. (43), $e^{-V\tau \operatorname{Tr} E^2}$, where E are the eigenvalues of H. This term favors a small value of λ. There is also a term due to the integral over E_μ for $\mu = k+1 \cdots V$. The requirement that these eigenvalues all be positive, and the effect of level repulsion between these eigenvalues and the first k eigenvalues, tend to favor a larger value of λ.

For example, in the simple non-interacting problem of the previous section, when we removed one eigenvalue, the level repulsion and the Gaussian weight canceled, and the eigenvalue could have been found anywhere between $-\sqrt{1/\tau}$ and $\sqrt{1/\tau}$. In the case considered in this section, the effects of level repulsion will be stronger, as k eigenvalues will coalesce. Further, the positivity constraint must be satisfied. We will also find from this a slight difference between λ for the large-N model and λ for the spherical model, which will imply a shift in energy from Eq. (9).

The integral over the upper sector of eigenvalues is performed in Appendix A. There, it is shown that the eigenvalue distribution of the upper sector is slightly distorted from the Wigner semicircle, and that λ is slightly shifted from $1/\sqrt{\tau}$ to $1/\sqrt{\tau} - \frac{3}{4}(1/\sqrt{\tau})\, 2^{1/3}(k/V)^{2/3}$. If the eigenvalue distribution of the upper sector were unchanged from a Wigner semicircle and $\lambda = \sqrt{1/\tau}$, the determinant $(\det(H_> + \lambda))^k$ would equal

$$\left(\frac{1}{2\sqrt{\tau}}\right)^{k(V-k)} e^{k(V-k)/2} \tag{53}$$

Also, $e^{-kV\tau\lambda^2}$ would equal e^{-kV}. However, the slight change in the eigenvalue spectrum, and slight shift in λ lead to

$$(\det(H_> + \lambda))^k \, e^{-kV\tau\lambda^2} = \left(\frac{1}{2\sqrt{\tau}}\right)^{k(V-k)} e^{k(V-k)/2} e^{-kV} \left(\frac{k}{V}\right)^{-(2/3)(k^2/2)} \quad (54)$$

as can be shown using the results in Appendix A and calculating the determinant by integrating the eigenvalue density from Appendix A.

In Appendix A we calculate λ and a, where a is the lowest eigenvalue of $H_>$. The sum, $\lambda + a$ gives a gap to excitations. The gap is equal to $\frac{1}{4} 2^{1/3} (1/\sqrt{\tau})(k/V)^{2/3}$ and will vanish as a power law in V. In Fig. 3 we illustrate the changed eigenvalue distribution, superimposing the Wigner semicircle of the original matrix H for comparison. The gaps in the figure are exaggerated and are not to scale; the figure is approximate only.

The factor of $\frac{2}{3}$ in the various above equations should not be a surprise. For a matrix with a Wigner semicircle distribution of eigenvalues, there is a square-root singularity in the eigenvalue density near the end of the semicircle. This means that the lowest k eigenvalues range in energy up to a distance of order $(k/V)^{2/3}$ from the end of the semicircle. Having changed the distribution of eigenvalues from the Wigner semicircle so that the lowest k eigenvalues all lie at zero energy, the eigenvalue density of the rest of the matrix will adjust, but once one looks at energies several times $(k/V)^{2/3}$ from the tail, the change in the eigenvalue distribution is small as a result of screening. This argument gives the scaling of the gap.
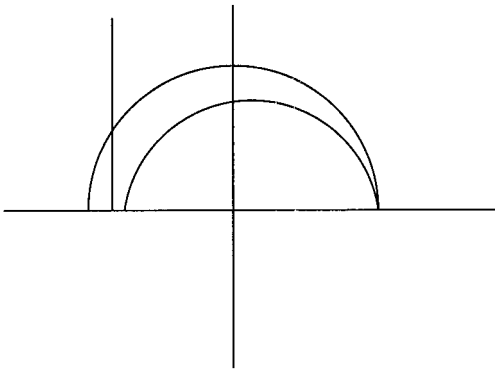

Fig. 3. Altered level distribution of $H + \lambda$ in large-N system. The spike contains k eigenvalues, the smaller semicircle contains $V - k$ eigenvalues. Superimposed is a semicircle of a matrix containing V eigenvalues. The horizontal axis is energy, although the scale is shifted so that the spike is at zero energy.

To understand the difference between Eq. (53) and Eq. (54), we must calculate $e^{-kV\tau\lambda^2}(\det(H_> + \lambda))^k$, for given λ. The naive evaluation of this is what gave Eq. (53). The evaluation of the determinant is very much like evaluating the potential energy of a system of charges interacting with a logarithmic Coulomb interaction. We can then write the true eigenvalue distribution of $H_>$ as the sum of two pieces: a Wigner semicircle extending from $-\sqrt{1/\tau}$ to $\sqrt{1/\tau}$, and a negative term extending to a distance of order $(k/V)^{2/3}$, with the integral of the negative term being k. The negative term can be thought of as a "hole."

The interaction with the Wigner semicircle, multiplied by $e^{kV\tau\lambda^2}$, is independent of λ, for λ inside the semicircle, and the product is equal to Eq. (53). So, we can think of Eq. (54) as arising from Eq. (53), multiplied by a contribution arising from the interaction of the k levels of H at $-\lambda$ with the "hole." This gives Eq. (54).

Finally, since we are considering the integral over $H_>$, recall that in Eq. (41), we have normalized the integral by

$$\frac{1}{\int dH \, e^{-V\tau \, \mathrm{Tr}(H^2)}} \tag{55}$$

In the calculation of Eq. (54) and of Appendix A, when we integrate over $H_>$, we gain an extra factor of

$$\int dH_> \, e^{-V\tau \, \mathrm{Tr}(H^2)} \tag{56}$$

The ratio of these produces a factor of

$$\sqrt{\frac{V\tau}{\pi}}^k \sqrt{\frac{2V\tau}{\pi}}^{kV - k^2/2 - k/2} \tag{57}$$

This is similar to the calculation in Section X.

B. Lower Sector

The rest of Eq. (41) involves the lower sector of eigenvalues. Let us write

$$\phi_i^\mu = v_i^\mu \sqrt{\frac{1}{E_\mu}} \tag{58}$$

Here, ϕ_i^μ is a ground state of the system, up to a change in normalization (compare to Eq. (11)). Further, let us change variables and write $E_\mu = \rho_\mu^{-1}$. Then, the lower sector of the integral can be written as

$$
\int d\rho_\mu \, d\phi_i^\mu \prod_\mu \rho_\mu^{-V/2 - 1 - (k-1)/2} \, \delta\left(\sum_\mu (\phi_i^\mu)^2 - \beta\right) \delta\left(\sum_i (\phi_i^\mu)^2 - \rho_\mu\right)
$$

$$
\times \prod_{\mu < \nu} \delta\left(\sum_i (\phi_i^\mu \phi_i^\nu)\right) |\rho_\mu - \rho_\nu| \det(M_{ij}) \tag{59}
$$

If we ignore the determinant in Eq. (59), the integral over ϕ, at fixed ρ, can be performed using a series of Lagrange multipliers to implement the δ functions. We can take the integral

$$
\int d\phi_i^\mu \prod_\mu \delta\left(\sum_\mu (\phi_i^\mu)^2 - \beta\right) \delta\left(\sum_i (\phi_i^\mu)^2 - \rho_\mu\right) \prod_{\mu < \nu} \left(\delta\left(\sum_i (\phi_i^\mu \phi_i^\nu)\right)\right) \tag{60}
$$

and write it as

$$
\int d\phi_i^\mu \, \frac{dc_{\mu\nu}}{2\pi} \frac{dc_\mu}{2\pi} \frac{dc_i}{2\pi} \, e^{\phi_i^\mu (ic_i \delta_{\mu\nu} + ic_{\mu\nu} + ic_\mu)\, \phi_i^\nu - i\rho_\mu c_\mu - i\beta c_i} \tag{61}
$$

We can integrate over the fields ϕ_i^μ, to obtain an action for the Lagrange multiplier fields c. The integral over c can then be handled by the saddle point method. One possible saddle point is $c_{\mu\nu} = 0$, $c_\mu = iE_\mu V/2$, $c_i = 0$. There is a zero mode for fluctuations about this saddle point which enforces the constraint $\sum_\mu \rho_\mu = \beta V$. This zero mode is a mode in which all the c_i are increased by dome constant, while all the c_μ are decreased by the same constant so that $c_\mu + c_i$ is unchanged for all μ and i.

At the saddle point, we find that the integral given by Eq. (61) is equal to

$$
e^{Vk/2} \prod_\mu \left(\frac{2\rho_\mu \pi}{V}\right)^{V/2} \tag{62}
$$

For V much greater than k, we can look at other fluctuations about the saddle point in a Gaussian approximation. The fluctuations in $c_{\mu\nu}$ contribute a factor of

$$
\int \frac{dc_{\mu\nu}}{2\pi} e^{-(1/2V) c_{\mu\nu}^2 \rho_\mu \rho_\nu} = \left(\frac{2\pi}{V}\right)^{+k(k-1)/4} (2\pi)^{-k(k-1)/2} \prod_\mu (\rho_\mu/V)^{-(k-1)/2} \tag{63}
$$

The fluctuations in c_i contribute a factor of

$$\int \frac{dc_i}{2\pi} e^{-(1/4) c_i^2 \sum_\mu (2\rho_\mu/V)^2} = \left(\frac{4\pi}{\sum_\mu (2\rho_\mu/V)^2} \right)^{V/2} (2\pi)^{-V} \qquad (64)$$

The fluctuations in c_μ can be ignored, as there are only k such terms fluctuating, and this would only lead to corrections to S_k of order k. We are not considering terms in S_k that small.

We will now evaluate the contribution to Z_k from $\det(M_{ij})$. This determinant involves both the lower and upper sector of eigenvalues. As discussed in Section IX, this matrix has $k(k+1)/2$ eigenvalues of order β^2 and $(V - k(k+1)/2)$ eigenvalues of order β. The trace of the matrix is $V\beta^2$. So, the $k(k+1)/2$ largest eigenvalues can be at most of order $V\beta^2/(k(k+1)/2)$. Let us assume that in fact each eigenvalue is approximately $V\beta^2/k^2$; making the approximation only leads to errors of order k^2 in S_k.

Since we will find in the end that $k \propto V^{2/5}$, we have $V - k(k+1)/2 \gg k(k+1)/2$. So, to find the product of the other $V - k(k+1)/2$ eigenvalues, we can simply throw out terms in M_{ij} of order β^2 (these are terms shown in Eq. (44)) and calculate the determinant of the remaining V-by-V matrix. We find

$$G_{ij} G_{ji} = 2 \sum_{\mu=1}^{k} \sum_{a=k+1}^{V} \frac{1}{E_\mu E_a} v_i^\mu v_j^\nu v_i^a v_j^a = 2 \sum_{\mu=1}^{k} \sum_{a=k+1}^{V} \frac{1}{E_a} \phi_i^\mu \phi_j^\mu v_i^a v_j^a \qquad (65)$$

Here, E_μ is of order $1/\beta$, while E_a is from the upper sector of eigenvalues. Averaging over ϕ_i^μ, we can ignore terms in which $i \neq j$ and obtain

$$G_{ii} G_{ii} \approx 2 \sum_{a=k+1}^{V} \beta \frac{1}{E_a} v_i^a v_j^a \qquad (66)$$

This is a Green's function of the matrix $H_>$. The Green's G_{ii} function of $H_>$ is approximately $2\sqrt{\tau}$ and so Eq. (66) is equal to

$$4\beta \sqrt{\tau} \qquad (67)$$

Putting these together, we obtain

$$\det(M_{ij}) \approx \left(\frac{V\beta^2}{k(k+1)/2} \right)^{k(k+1)/2} (\beta 4 \sqrt{\tau})^{V-k(k+1)/2} \qquad (68)$$

There will be corrections to this which lead to corrections to S_k of order $V/\sqrt{k}$. These corrections will be discussed at the end of this section and in Appendix B.

We now need to know what the spectrum of E_μ is, for $\mu = 1 \cdots k$, to determine what some of the contributions calculated above are equal to. The exact spectrum is difficult to determine, but we can determine enough for our purposes. There is an effect of level repulsion which tends to push the energies apart, but it may be seen that the other terms in Eq. (59) are greatest when the energies are all equal to $k/(V\beta)$. The effects of level repulsion will push the E_μ apart, but on the other hand if we considered only the effects of level repulsion, ignoring all other effects except the requirement that all eigenvalues be positive, we still want all E_μ to be of order $k/(V\beta)$, up to factors of order unity. So, considering all effects, we must have that E_μ is of order $k/(V\beta)$.

Also, we would like to know how large $|E_\mu - E_\nu|$ is. Although each E_μ is of order $k/(V\beta)$, the separation between energy levels may be much smaller, as the contribution of Eq. (64) depends on E_μ and is greatest when all E_μ are equal. There are other terms which have a similar effect and are also greatest when all E_μ are equal. However, we will now show that despite this $|E_\mu - E_\nu|$ is still of order $k/(V\beta)$. It can be shown that, if Eq. (64) is expanded out in $\delta E_\mu = E_\mu - k/(V\beta)$, that there will be term of the form

$$e^{-(1/4)\,V(\delta_\mu/\beta V)^2} \tag{69}$$

Thus, the energy levels E_μ repel each other due to the absolute value, $|E_\mu - E_\nu|$, but are confined by a potential that, for E_μ approximately equal to $k/(V\beta)$, is given by Eq. (69). If we have k levels, E_μ, repelling each other, in a confining potential given by Eq. (69), then for k of order $V^{1/3}$, or greater, we find that the separation between levels is of order $k/V\beta$. So, we can assume that $|E_\mu - E_\nu|$ is of order $k/(V\beta)$ since we will find in the end that $k = V^{2/5} > V^{1/3}$.

Putting all this together, we obtain a contribution to Eq. (41) from the lower sector given by Eq. (59) and a contribution from the upper sector given by Eqs. (54), (57), as well as the determinant of Eq. (68). All this is at fixed λ. Equation (59) includes a number of terms due to the change of variables and from the level repulsion, as well as contributions evaluated in Eqs. (62)–(64). If we take typical ρ_μ to be of order $V\beta/k$ as discussed in the above paragraphs on level repulsion, we find

$$Z_k \approx \left(\frac{V}{k^2}\right)^{k^2/2} \left(\frac{k}{V}\right)^{+(1/3)(k^2/2)} \left(\frac{4k\tau}{\pi}\right)^{V/2} \tag{70}$$

C. Site to Site Fluctuations in λ_i

The above calculation was performed for given λ. Finally, we must obtain the strength of fluctuations in λ, and perform the integration over λ.

We will now obtain an effective action for λ. Let $\lambda_i = \lambda + \delta\lambda_i$, where λ is the average value of λ_i calculated above. Then, $\sum_i \delta\lambda_i = 0$; this is important and will be discussed more below. The action for λ_i arises from the change in $(\det(H_> + \lambda))^k$, as well as the change in H required to ensure that $H + \lambda$ still has k zero eigenvalues.

First, let us evaluate the change resulting from the determinant. We have

$$(\det(H_> + \lambda))^k = e^{k \operatorname{Tr} \log(H_> + \lambda_i)} \tag{71}$$

Expanding the log in the above equation in $\delta\lambda_i$ we obtain to second order

$$e^{k \operatorname{Tr} \log(H_> + \lambda)} e^{-k(1/2) G_{ii}^2 \delta\lambda_i^2} = e^{k \operatorname{Tr} \log(H_> + \lambda)} e^{-2k\tau\delta\lambda_i^2} \tag{72}$$

Finally, remember that $H + \lambda_i$ must have k zero eigenvalues. For $\lambda_i = \lambda$, a constant, this contributed a term $e^{-kV\tau\lambda^2}$ as found in Eq. (54). Let us write H in a basis of the eigenvectors of $H + \lambda_i$. Then, since $(H + \lambda_i) v^\mu = 0$, for $\mu = 1 \cdots k$, we must have $H_{\mu, a} = \sum_i v_i^\mu \lambda_i v_i^a$, for $\mu = 1 \cdots k$ and $a = 1 \cdots V$. Also $H_{a, \mu} = \sum_i v_i^a \lambda_i v_i^\mu$. Then, we must evaluate

$$e^{-V\tau \sum_{\mu=1}^k \sum_{a=1}^V (H_{\mu, a}^2 + H_{a, \mu}^2)} = e^{-2V\tau \sum_{\mu=1}^k \sum_{a=1}^V (\sum_i v_i^a \lambda_i v_i^\mu)^2} \tag{73}$$

Since $(v_i^\mu)^2$ and $(v_i^a)^2$ are approximately $1/V$, this is

$$e^{-2k\tau\delta\lambda_i^2} \tag{74}$$

Multiplying Eqs. (72) and (74), and integrating over λ_i, we get find an extra contribution to Eq. (41)

$$\left(\frac{\pi}{4k\tau}\right)^{V/2} \tag{75}$$

Corrections to this will lead to corrections to S_k of order $V/\sqrt{k}$. These will be discussed later.

The result from Eqs. (72), (74) is interesting in that it requires that $\sum_i \delta\lambda_i = 0$. If we evaluated the shift in determinant for constant shift in λ (all $\delta\lambda_i$ equal to the same constant), it would exactly cancel the change in $e^{-V\tau H^2}$. However, here we would find that the perturbative method used to obtain Eq. (72) was not valid, and that constant shift in λ could violate Eq. (8). Further, we would not have the factor of two in Eq. (74) for constant shift in λ. The factor of two arises since the shift due to $\delta\lambda_i$ involves off-diagonal terms in H, while a constant shift in λ would involve diagonal

terms in H. Effectively, if we look at fluctuations in λ_i, the mode in which all λ_i fluctuate together is very different from the other modes in which $\sum_i \delta\lambda_i = 0$. The mode in which all λ_i fluctuate together will be discussed in the next section; this mode is important for considering fluctuations in total energy.

D. Results

The final result, combining Eqs. (70) and (75), is that Z_k is of order

$$\left(\frac{V}{k^2}\right)^{k^2/2}\left(\frac{k}{V}\right)^{+(1/3)(k^2/2)} \tag{76}$$

In obtaining this result a large number of terms have canceled. It may be verified that all terms with an exponent of order kV or V cancel. Further, it may be verified that τ cancels exactly, as required.

Equation (76) is maximized for

$$k \propto V^{2/5} \tag{77}$$

There will also be terms contributing to Z_k which look like e^{ak^2}, for some constant a, as well as exponential of lower powers of k^2 and powers of V. To evaluate these terms is beyond the scope of the present calculation. These terms can shift the constant of proportionality in Eq. (77), but cannot change the power law dependence of k on V. It would be nice if a more detailed calculation could obtain the exact proportionality constant, as well as checking that $\sum_k Z_k = 1$. The requirement that $\sum_k Z_k = 1$ can be verified here only so far as noting that for $k \propto V^{2/5}$ that Z_k does not have any terms in it like $e^{k^2 \log k}$ or like e^V, but we cannot check for cancellation of contributions to S_k of order $k^2 = V^{4/5}$ or lower order.

Further, there will be contributions to S_k of order $V/\sqrt{k}$ that would arise from a more careful calculation of $\det(M_{ij})$ and from a better calculation of the integral over fluctuations in λ_i. We will discuss these terms in Appendix B, and show that they are in fact of order $V/\sqrt{k}$. For $k = V^{2/5}$, these terms will be of the same order as the k^2 contributions to S_k. For $k < V^{2/5}$, these terms will be the most important contribution to S_k, and since they are negative, we will find that S_k is negative and Z_k is much less than 1; for $k > v^{2/5}$, Eq. (76) will give the most important contribution and will also lead to Z_k being small. Therefore, although terms of order $V/\sqrt{k}$ can change the proportionality constant in Eq. (77), they cannot change the power law dependence of k on V. These terms can also lead to a change in the gap, $\lambda + a$, derived in Appendix A, but will not change the scaling of the gap with V.

The results on the ground state can be summarized by Eq. (77) for the value of k, Eq. (74) for the fluctuations in λ_i, and the calculation in Appendix A for the average value of λ, and the weak gap to excitations above the ground state.

XII. DISCUSSION OF RESULTS ON THE GROUND STATE AND CONNECTION TO PROPERTIES IN THE SPIN GLASS PHASE

The two most important results are the value of k and the strength of fluctuations in λ. We will discuss these results in this section and try to interpret them. Next we will discuss fluctuations in the total energy of the system, as discussed at the end of Section XI, Subsection C. The shift in λ is also interesting all will be discussed next. Finally, we will argue that the results of the previous section are also applicable to the spin glass phase as well as the ground state.

The value of k is perhaps somewhat suprising. One might have expect $k \propto \sqrt{V}$ as we have the bound that $k(k+1)/2 \leqslant V$. However, imagine considering a different problem from that of finding ground states of the system. Suppose we had instead looked for stationary states. These are configurations of the spins such that the energy does not change for small changes in spin configuration. The problem of finding stationary states is the same as the problem of finding ground states, except that Eq. (8) does not need to be satisfied any more. Then, consider a solution in which all λ_i are equal to some constant λ. Now, we can take λ to lie in the middle of the Wigner semicircle, instead of near the edge. Then, the solution of the problem would proceed exactly as before, except that the corrections to the determinant calculated in Appendix A would be of order

$$\left(\frac{V}{k}\right)^{k^2/2} \tag{78}$$

without the factor of 2/3 in the exponent (compare to the discussion of the origin of the factor 2/3, as this factor arises near the edge of the semicircle). In this case, it may be verified that Z_k is greatest for k^2 of order V.

Since the same bound on k holds for stationary states as for ground states, the most likely value of k for stationary states is not surprising, as here k^2 is of order V. However, one can perhaps argue that typical k should be less for a ground state than for a stationary state. From most stationary states one can reduce the energy by aligning some spins and reducing k, bringing one closer to the ground state. This argument is of

course very crude, but it is an attempt to interpret the result for k, especially since the calculation of Section XI and Appendix A shows that the result for k depends crucially on the square-root singularity of the density of states of H.

In fact, it is not surprising that the square-root singularity should lead to a reduction in k. We expect that λ will be near $\sqrt{1/\tau}$. The system must take k eigenvalues near $-\sqrt{1/\tau}$ and bring them to zero energy, by tuning λ_i. The square-root singularity implies that there is a lower density of eigenvalues near the end of the semi-circle, and so it is more difficult to bring k eigenvalues to zero energy. When looking for stationary states, instead of ground states, we can take λ near the middle of the semicircle, and there are many more eigenvalues around to bring to zero energy, so k can be bigger.

More interesting is the strength of fluctuations in λ. We have shown that the mean square fluctuations are of order $1/k$. The fluctuations in λ on a given site measure the fluctuations in the energy of the bonds connecting that site to other sites. Each site is connected to V other sites. The term in the Hamiltonian connecting the two sites has mean square of order $1/V$. However, the energy due to a bond connecting two sites is determined both by the strength of the bond and the correlation between the two spins on the sites. Above, we estimated that the mean square correlation function between two sites is of order $1/k$. Putting all this together, the energy on a given site is the sum of V terms, each with mean square fluctuations of order $1/Vk$. So, it is not surprising that the mean square fluctuation in λ for a given site would be of order $1/k$.

The total ground state energy of the system is given by the sum of the different λ_i. Since $\sum_i \delta\lambda_i = 0$, we must look at fluctuations in λ to get the fluctuations in the ground state energy. From the calculation of Appendix A, and from the discussion of the "hole" in Section XI, we understand that λ is being attracted to this "hole" which has width $(k/V)^{2/3} = k^{-1}$. If λ increases beyond $\sqrt{1/\tau}$ then $e^{-kV\tau\lambda^2}(\det(H_> + \lambda))^k$ becomes small since the contribution to the determinant from the semi-circle decreases rapidly for $\lambda > \sqrt{1/\tau}$. On the other hand, if λ gets much smaller than the value calculated in Appendix A, $\lambda = -\sqrt{\tau} + \frac{3}{4}\sqrt{\tau}\, 2^{1/3}k^{2/3}$, it will be difficult to satisfy Eq. (8). So, λ varies over an interval of range k^{-1}. As λ is varied over this interval, one must find how $e^{-kV\tau\lambda^2}(\det(H_> + \lambda))^k$ varies; this can be accomplished by a calculation along the lines of that in Appendix A, or by using the following simple argument. The attraction between λ and the "hole" is effectively a logarithmic Coulomb attraction between two charges of opposite sign and magnitude k. The hole has width k^{-1}. A simple calculation on this

Coulomb system shows that λ can vary only over a region of width k^{-2}, where one factor of k^{-1} arises from the width of the "hole" and the other factor of k^{-1} arises from the magnitude of the charges. So, the fluctuation in the energy of the system is proportional to $Vk^{-2} = V^{1/5}$ and the square fluctuation in energy is proportional to $V^{2/5}$.

This may at first seem surprising. For finite dimensional random systems, the square fluctuation in the energy is usually of order V by the central limit theorem; the total energy is the sum of V different quantities with fluctuations of order unity. However, in an infinite range model such as is considered here, each site is connected to V different sites, and the fluctuations in the energy for a given site are correspondingly much smaller. Therefore, it is possible for the mean square fluctuation in total energy to be of order $V^{2/5}$ instead of V.

In the paramagnetic phase, where the mean-square fluctuations in λ_i are of order $1/V$, the mean-square fluctuation in total energy is of order unity.

The shift in λ, leading to λ slightly greater than $-\sqrt{1/\tau}$ is not surprising. This implies that the ground state energy of the large-N system is slightly higher than that of the spherical model. This is of course not surprising since the spherical model has additional freedom to choose its ground state, as the constraint on the sum of the spins from the spherical model is less constricting than the large-N constraint on the length of each individual spin.

Having obtained these results for the zero temperature problem, we expect that they will hold true for the spin glass phase. As discussed above, the thermodynamic limit and large β limit both involve sending eigenvalues to zero, so the properties of the spin glass phase should be very similar to the zero temperature problem.

XIII. CONNECTION WITH EFETOV SUPERSYMMETRY

We will discuss a slight modification to the previously constructed supersymmetric formalism. This modification will make the formalism very closely related to the Efetov supersymmetry used in calculations on non-interacting disordered systems. It opens the possibility of using field theory techniques to deal with disordered and glassy N systems in the large-N limit; this might prove much more useful for finite dimensional systems when the techniques of random matrix theory are not available. Unfortunately, in the spin glass phase of the infinite range model it has not yet been possible to proceed with this modified formalism to the extent that has been done above with the original formalism. Still, the technique is

interesting in itself, and will provide a field-theoretic method of obtaining the properties of the paramagnetic phase discussed above.

One complication when trying to apply field theory techniques to Eq. (38) is the appearance of the Green's function in the exponential. This makes it impossible to do what one would normally like to do, namely integrate over H before integrating over any other variables, as it is difficult to integrate over matrices H_{ij} when the action depends on the inverse of the matrix. We can rewrite Eq. (38) in a slightly simpler form that avoids some of these complications. First we can rewrite it as

$$\int_{G \geqslant 0} dG_{ij} \, d\lambda_i \, \frac{dc_i}{2\pi} \, d\bar{a}_i \, da_i \, e^{ic_i(G_{ii}-1)} e^{\bar{a}_i G_{ij} a_j G_{ji}} \delta(G - (H+\lambda)^{-1}) \qquad (79)$$

We can replace the δ-function by $\delta((H+\lambda)\,G-1) \det(H)^V$. Writing the δ-function by an integral over a set of auxiliary variables c_{ij} and writing the determinant as an integral over a set of Grassman variables $\bar{\psi}_{ij}, \psi_{ij}$, we obtain

$$\int_{G \geqslant 0} dG_{ij} \, dc_{ij} \, d\bar{\psi}_{ij} \, d\psi_{ij} \, d\lambda_i \, dc_i \, d\bar{a}_i \, da_i \, e^{ic_i(G_{ii}-1)} e^{\bar{a}_i G_{ij} a_j G_{ji}} e^{i \operatorname{Tr}(c(HG-1))} e^{\operatorname{Tr}(\bar{\psi} H \psi)} \tag{80}$$

This introduces a second supersymmetry into the problem, connecting the variables c, G with the variables $\bar{\psi}$, ψ. This supersymmetry is closely related to the Efetov supersymmetry used in calculations for disordered system.[6]

In the paramagnetic phase, it is possible to use this formalism very simply. It has not yet been possible to obtain anything interesting in the spin-glass phase using this formalism. In the paramagnetic phase, we can integrate over H, and then introduce a supermatrix field Q to decouple the integral, following procedures similar to those used in the Efetov super-symmetry technique.[6] In the paramagnetic phase, a simple saddle point approximation on Q suffices, since Q has a gap for fluctuations. At this point, the notation becomes quite complicated, although the ideas are simple, and not very different from those used in the Efetov technique used for non-interacting systems. We will simply state that such a saddle-point technique reproduces all the results of the section on the paramagnetic phase of the large-N spin glass.

In the spin-glass phase, the fluctuations of the superfield Q become gapless, and it becomes difficult to proceed with this technique. We expect that the bosonic sector of Q may have k eigenvalues separate out from the others; there will be then $V-k$ remaining bosonic components and V remaining fermionic ones. The difference of k between these may produce

the factor of $(\det(H_> + \lambda_i))^k$ that was so crucial before. However, it is difficult to use Efetov supersymmetry near the tail of the spectrum of eigenvalues of a random matrix. This is the problem in proceeding further in the formalism of this section.

XIV. TROUBLES WITH REPLICA SOLUTIONS TO THE PROBLEM

We will briefly review the replica solution to the large-N spin glass problem,[3] indicating a number of mathematical problems. The replica solution yields good results in the paramagnetic phase, but does not correctly address the spin glass phase. Two important questions considered in this paper, namely the number of spin components used to form the spin glass state and the site-to-site or sample-to-sample fluctuations in λ, are not even considered within this replica formalism. However, we can point, out a few other mathematical problems.

Since the value of k, and the strength of fluctuations in λ, are not addressed in any previous replica work on this problem, the results of that work must be incomplete. Within the paramagnetic phase, the results of that work, as far as they go, are accurate. Similarly for the phase transition. However, within the spin glass phase, we have argued in Section VIII that the correlation functions of the system depend on k. If one instead extracts correlation functions from the mean-field Edwards-Anderson order parameter of the previous replica work, one obtains the wrong result for correlation functions. The purpose of this section is to show the mathematical problems responsible for this wrong result: the main problem is the large number (N^2) of components of the decoupling matrix, and the integration over these components.

In the replica technique we compute a partition function such as

$$Z = \int_{\sum_\mu (\phi_i^{\mu, \alpha})^2 = N} d\phi_i^{\mu, \alpha} e^{\beta H} \tag{81}$$

where ϕ_i^μ has now been given an extra index α. This is a replica index ranging from $1 \cdots n$. In the end, the limit $n = 0$ is taken.

After averaging over H, one can then decouple the average with a replica matrix $Q_{\alpha, \beta, \mu, \nu}$. The result is

$$Z = \int_{\sum_\mu (\phi_i^{\mu, \alpha})^2 = N} d\phi_i^{\mu, \alpha} \, dQ_{\alpha, \beta} \, e^{-V\tau \, \mathrm{Tr}(Q^2)} e^{\phi_i^{\mu, \alpha} Q_{\alpha, \beta, \mu, \nu} \phi_i^{\nu, \beta}} \tag{82}$$

Finally, the constraint on the length of the spins is enforced with a Lagrange multiplier λ_i^α to obtain

$$Z = \int_{\sum_\mu (\phi_i^{\mu,\,\alpha})^2 = N} d\lambda_i^\alpha \, d\phi_i^{\mu,\,\alpha} \, dQ_{\alpha,\,\beta} \, e^{-V\tau \operatorname{Tr}(Q^2)} e^{\phi_i^{\mu,\,\alpha}(Q_{\alpha,\,\beta,\,\mu,\,\nu} + i\lambda_i^\alpha)\,\phi_i^{\nu,\,\beta}} e^{-iN\lambda_i^\alpha} \qquad (83)$$

Taking saddle points in both Q and λ, one can obtain results in the paramagnetic phase. However, there are several problems. First of all, the decoupling matrix has a number of components of order N^2. The fact that Q has a gap to fluctuations in the paramagnetic phase of order V is not very useful in the large-N limit, since the number of components is so great. It is interesting to note that restricting N to be of order $\sqrt{V}$, remembering the result on the maximum k to form a ground state, we would find that the matrix Q must still have of order V different components.

The fluctuations in Q do have at least one important effect. If one ignored these fluctuations, one would naively think that the gap for fluctuations in λ_i is of order N, since for fixed Q this is indeed the correct result. However, integrating over the N^2 different components of Q must reduce the gap for fluctuations in λ_i to be of order V, as this is the result found in Section VII.

Another problem has to do with the spin glass phase. In the supersymmetric formalism, we had to restrict to the sector $H + \lambda_i \geqslant 0$. This might seem to be a problem with the supersymmetric formalism, as this is a slightly strange restriction to enforce. However, a similar problem will arise in the replica formalism. The procedure used is to integrate over all H and then decouple this to produce an integral over Q. Then, an effective action for Q and λ is obtained by integrating over ϕ. However, since the fields ϕ are bosonic, the integral over ϕ is only well defined if $H + \lambda \geqslant 0$. So, the procedure of integrating over H, decoupling, and then integrating over ϕ, is ill-defined, unless one also can somehow apply the same restriction $H + \lambda_i \geqslant 0$. One cannot integrate over all H at fixed λ and still be assured that $H + \lambda$ will be positive definite.

Further, in the spin glass phase, we saw from the previous sections that $H + \lambda$ is close to gapless; the gap vanishes as a power law in the thermodynamic limit. The replica matrices also will have gapless excitations. Considering the calculation of Appendix A, we expect that a detailed consideration of fluctuations is crucial to getting correct answers here, and cannot easily be obtained by using replica techniques, or techniques outlined in the Section XIII.

Another problem lies in the order of limits. There are the limits $n \to 0$, $N \to \infty$, and $V \to \infty$. Even without the replica limit, there are complications

in interchanging the large N and large V limits, as we have found in this paper.

In conclusion, although the replica technique is very powerful for certain problems, especially problems with interaction and finite N, where other techniques cannot be used, the old results in the literature for the large N spin glass using replica techniques should not be trusted fully. Although very useful and enlightening in the paramagnetic phase, in the spin glass phase there will be problems.

XV. HERMITIAN AND SYMPLECTIC SYSTEMS

Instead of the system of real Hamiltonians considered in above, we can consider more general systems. One possibility it to take the spins ϕ to be N component complex vectors, instead of N component real vectors. Then, we can take the matrix H_{ij} to be an arbitrary Hermitian matrix instead of simply a real symmetric matrix. So, the system will be defined by the Hamiltonian

$$H = \sum_{i, j, \mu} \bar{\phi}_i^\mu \phi_j^\mu H_{ij} \tag{84}$$

where i, j index the various sites and range from 1 to V, while μ indexes various components of the spin and ranges from 1 to N. We have the constraint that

$$\sum_\mu |\phi_i^\mu|^2 = N \tag{85}$$

We will simply sketch the changes in the Hermitian case from the real case considered above. Previously, we derived the bound that $k(k+1)/2 \leqslant V$ where k was the number of spin components needed to form a ground state for a given H. This was based on the fact that for a real matrix $H + \lambda$, to obtain k zero eigenvalues requires $k(k+1)/2$ free parameters. For a general Hermitian matrix $H + \lambda$, obtaining k zero eigenvalues requires k^2 parameters. So, we may obtain the bound that $k^2 \leqslant V$. Here, k counts the number of complex spin components.

Similarly, a sypersymmetric formalism may be derived for the Hermitian case. The formalism of Section IX requires works equally well for the Hermitian case. The only change required is that we will write $H + \lambda = U^\dagger E U$, instead of $H + \lambda = O^{\mathrm{T}} E O$. Then, the measure of integration will be different in the Hermitian case, so that we will write $\prod_{\mu < \nu} |E_\mu - E_\nu|^2 \, |\det(H_> + \lambda)|^{2k}$, instead of $\prod_{\mu < \nu} |E_\mu - E_\nu| \, |\det(H_> + \lambda)|^k$. The calculation of Section XI

can still be pursued for the Hermitian case, and the power law dependences will be the same in the Hermitian case as in the real case considered above.

The formalism of Section XIII must be slightly changed for the Hermitian case. The matrices G and H can now be complex. To implement the constraint $\delta(HG - 1)$ we will then need to use a complex field c. Using complex G and c amounts to doubling the number of bosonic degrees of freedom. We will then need to use two fields $\bar{\psi}$ and two fields ψ to get the correct Jacobian. However, all these changes are easy to implement.

There is also a symplectic case, in which H is taken from the symplectic ensemble of matrices. In the symplectic case, we have the bound $2k^2 - k \leqslant V$. Also, the level repulsion factor in Section IX must be replaced with $\prod_{\mu < \nu} |E_\mu - E_\nu|^4 \, |\det(H_> + \lambda)|^{4k}$. Similarly, compared to the real case, the symplectic case will require four times as many fields using the formalism of Section XIII. These changes are also all easy to implement.

XVI. CONCLUSION

In conclusion, we have looked at the problem of the large N infinite range spin glass, and obtained several results on the spin glass phase. One reason for the interest in the problem is that it is perhaps the simplest system which combines both disorder and interaction. It is simple enough that one feels that there should exist a simple solution, but, as discussed above, previous attempts to solve this problem do not fully succeed. It is perhaps surprising that such a simple problem requires a technique as complicated as that used here.

This problem exhibits many interesting aspects of disordered systems. It has an analogue of Griffiths effects and a non-trivial ground state. The infinite range spin glass with finite N has been solved by replica techniques before. However, if is nice to have a glassy sytem that can be solved without using replica techniques, to use as a check. For curtain aspects of random matrix theory, such as level-level correlation functions, replica techniques run into troubles, so it is interesting to find a supersymmetric technique to deal with a glassy system.

The results derived include the scaling of k as $V^{2/5}$; the scaling of the gap to excitations as $(k/V)^{2/3} = V^{-2/5}$; and the magnitude of fluctuations in λ_i. It has not yet been possible to derive the exact ratio of $k/V^{2/5}$. It is interesting that we deal with many quantities which vanish in the thermodynamic limit in this paper. The gap, the ratio k/V, the correlation functions in the spin glass phase, and other quantities all vanish in the large V limit, but they do so more slowly than $1/V$, which makes them interesting.

The method introduced in the paper will hopefully be of more general use. It would be very interesting to try to use this technique to attack

problems with finite range interactions. Finite dimensional, large N spin glass systems are one possibility. Another possibility are finite dimensional, large N system with disorder, but without frustration. One example of these system, the large N dirty boson model,[4] has been treated with an RG technique. The supersymmetric technique outlined here might be useful in that context. It would also be very interesting if a method were found for including $1/N$ corrections in this formalism. For finite N in the large V limit we expect replica symmetry breaking, so it is not clear what would happen given the first $1/N$ correction to the large N limit. The self-consistency equation could have multiple solutions for finite N, and it is not clear what would happen.

APPENDIX A: SOLUTION OF VARIATIONAL EQUATION FOR λ

Here we consider a problem in which all λ_i are equal to each other. We will write $\lambda_i = \lambda$. The problem is to find the most likely value of λ. The reason for considering this problem is discussed at the start of Section XI.

Equation (40) depends on λ in two ways. First, there is the factor of $e^{-V\tau \operatorname{Tr} H^2}$. Second, there is a factor of $(\det(H_> + \lambda))^k$. As discussed at the start of Section XI, we must compute

$$e^{-kV\tau\lambda^2}(\det(H_> + \lambda))^k \tag{86}$$

Let $H_>$ have a density of eigenvalues equal to $V\rho(x)$. Since $H_>$ has $V - k$ eigenvalues, we will have $\int \rho(x)\, dx = (V - k)/V$. The problem of finding the density of eigenvalues of $H_>$ and the correct value of λ reduces to maximizing the functional

$$- Vk\tau\lambda^2 + V^2 \frac{1}{2} \int dx\, dy\, \rho(x)\, \rho(y) \log |x - y|$$

$$+ V^2 \int dx\, \rho(x) \left(-\tau x^2 + \frac{k}{V} \log |x + \lambda| \right) \tag{87}$$

where the first integral represent the effects of level repulsion and the second integral represents the effects of the Gaussian confining potential and the determinant $(\det(H_> + \lambda))^k$.

First, we will find the density of eigenvalues of $H_>$ for given λ. Then, we will find the optimal value of λ. By varying Eq. (87) with respect to $\rho(x)$, and then differentiating the resulting functional equation with respect to x, we obtain

$$\int dy\, \rho(y) \log |x - y| = 2\tau x - \frac{k}{V} \frac{1}{x + \lambda} \tag{88}$$

Equation (88) only holds for x such that $\rho(x) > 0$. We assume that $\rho(x)$ is non-zero for $a < x < b$, and zero for $x < a$ or $x > b$.

The techniques for solving this equation can be found in the book.[10] We simply quote and use the results of that work here.

The function $\rho(x)$ can be obtained from the expression expression

$$\rho(x) = \frac{1}{\pi^2} \sqrt{(x-a)(b-x)} \int_a^b \frac{dy \, 2\tau y - (k/V)(1/(y+\lambda))}{\sqrt{(a-x)(x-b)} \, (y-x)} \tag{89}$$

This integral can be performed by a series of simple steps: write $y = ((b-a)/2) z + (a+b)/2$ so that the integration range for z extends from -1 to 1. Then write $z = \cos(\theta)$ to get an integral over θ from 0 to π. Finally write $w = e^{i\theta}$, for a complex variable w and extend the integration range for θ from $0 \cdots \pi$ to $0 \cdots 2\pi$. Then, the integral over w is an integral over the unit circle in the complex plane and can be performed by contour integration.

The result is

$$\rho(x) = \frac{1}{\pi} \sqrt{(x-a)(b-x)} \left(2\tau + \frac{k}{V\sqrt{(a+\lambda)(b+\lambda)}} \frac{1}{\lambda+x} \right) \tag{90}$$

For $k = 0$, this reduces to the well known Wigner semi-circle.

There is also a consistency equation that must be satisfied that helps determine the limits a and b. This equation is

$$0 = \frac{1}{\pi^2} \sqrt{(x-a)(b-x)} \int_a^b \frac{dy \, 2\tau y - (k/V)(1/(y+\lambda))}{\sqrt{(a-x)(x-b)}} \tag{91}$$

This integral can be performed by the same contour integration techniques as the previous one. The result is

$$0 = \tau(a+b) - \frac{k}{V\sqrt{(a+\lambda)(b+\lambda)}} \tag{92}$$

In the case $k = 0$, this reduces to the requirement that $a + b = 0$.

Finally, we have the constraint that $\int_a^b \rho(x) \, dx = (V-k)/V$. This integral can also be performed using contour integration. The result here is

$$\frac{V-k}{V} = \tau \left(\frac{b-a}{2} \right)^2 - k + \frac{k}{V} \frac{(a+b)/2 + \lambda}{\sqrt{(b+\lambda)(a+\lambda)}} \tag{93}$$

For $k = 0$, this equation gives $b - a = [1/(2\sqrt{\tau})] \sqrt{(V-k)/V}$.

Putting Eqs. (92) and (93) together, we can obtain a and b. Let us write

$$b - a = \frac{2 + l}{\sqrt{\tau}} \tag{94}$$

$$\lambda = -a + \frac{\delta}{\sqrt{\tau}} \tag{95}$$

We assume that l and δ are small numbers. For $k = 0$, we will find that l is of order k/V. We approximate $\sqrt{(b + \lambda)(a + \lambda)} = \sqrt{2\delta}/\tau$. Then Eq. (92) becomes

$$a + b = \frac{k}{V} \frac{1}{\sqrt{2\delta}} \tag{96}$$

Ignoring terms of order k/V, or smaller, in Eq. (93), we find

$$0 = \sqrt{\tau}\, l + \frac{k}{V} \frac{\tau}{\sqrt{2\delta}} \tag{97}$$

Combining these we find that $a + b = -l/\sqrt{\tau}$, or $b = 1/\sqrt{\tau}$ up to corrections of order k/V.

Finally, we can derive an equation for λ. Varying λ and requiring that Eq. (87) be stationary gives

$$2\tau\lambda = \int_a^b dy\, \rho(x) \frac{1}{y + \lambda} \tag{98}$$

It is possible in general to evaluate the following integral

$$\int_a^b dy\, \rho(x) \frac{1}{y - x} \tag{99}$$

The result is

$$\sqrt{(\lambda + a)(\lambda + b)} \left(2\tau \frac{\lambda}{\sqrt{(\lambda + a)(\lambda + b)}} - 2\tau \right.$$
$$\left. + \frac{k}{V} \frac{1}{x + \lambda} \left(\frac{1}{\sqrt{(a - x)(b - x)}} - \frac{1}{\sqrt{(\lambda + a)(\lambda + b)}} \right) \right) \tag{100}$$

Taking a limit as x goes to λ in the above equation, and substituting into Eq. (98), we find

$$2\tau = \frac{(k/V)((a+b)/2 + \lambda)}{(\lambda + a)(\lambda + b)^{3/2}} \tag{101}$$

Finally, using the consistency and volume equations, as well as the equation for λ, we find that $b = \sqrt{\tau}$, $a = -1/\sqrt{\tau} + (1/\sqrt{\tau})\, 2^{1/3}(k/V)^{2/3}$, and $\lambda = 1/\sqrt{\tau} - (3/4)(1/\sqrt{\tau})\, 2^{1/3}(k/V)^{2/3}$.

APPENDIX B: CONTRIBUTION TO S_K OF ORDER $V/\sqrt{K}$

As discussed in Section XI, the calculation of the fluctuations in λ_i and the calculation of the determinant of M_{ij} is only approximate. There will be corrections to this calculation which will produce correction to S_k of order $V/\sqrt{k}$.

We will not precisely evaluate these terms, but we will show that they are of order $V/\sqrt{k}$; this is slightly surprising since there are many other corrections to S_k which are of order V/k. For example, the calculation of the integral over c_i, so that Eq. (64), which includes only Gaussian fluctuations in c_i is not quite right. If calculated more carefully, there would be corrections of order V/k from the cubic and quartic terms in the action c_i.

We will look carefully only at the corrections to order $V/\sqrt{k}$, the correction to the calculation of the determinant of M_{ij}. The calculation will only be sketched, to save space. Off-diagonal terms in M_{ij}, after removing terms in M_{ij} of order β^2 as discussed in Section XI, are equal to

$$2 \sum_{\mu=1}^{k} \sum_{a=k+1}^{V} \frac{1}{E_a} \phi_i^\mu \phi_j^\mu v_i^a v_j^a \tag{102}$$

These terms have mean square $(1/k)(G_{ij}^>)^2$, where $G_{ij}^>$ is the Greens function of matrix $H_> + \lambda$. If one writes the determinant as $e^{\mathrm{Tr}\,\log M_{ij}}$, and perturbatively expands the log in the off-diagonal terms, the first correction one finds is $e^{(1/2)\sum_{i,j=1}^{V}(1/k)(G_{ij}^>)^2}$. This might make one think that these corrections will only change S_k by order V/k. However, although in the paramagnetic phase $(G_{ij}^>)^2$ is of order $1/V$ this is not true in the spin glass phase, and the perturbative expansion diverges and is therefore not valid.

We can use a trick to get around this. The desired determinant can be written

$$\frac{\int d\bar{\psi}_a^\mu \, d\psi_a^\mu \prod_i \left(\sum_{\mu,\nu} 2\bar{\psi}_a^\mu v_i^a \phi_i^\mu \phi_i^\nu v_i^a \psi_a^\nu \right) e^{\bar{\psi}^\mu (H_> + \lambda)\, \psi^\mu}}{(\det(H_> + \lambda))^k} \tag{103}$$

Here, $\mu = 1 \cdots k$ and $a = k + 1 \cdots V$. The vectors v^a are the eigenvectors of $(H_> + \lambda)$. In the basis of these eigenvectors, $(H_> + \lambda)$ is diagonal. For simplicity, let us evaluate this with a specific choice of ϕ_i^μ. Let us take $\phi_i^1 = \beta$ for $i = 1 \cdots V/k$ and $\phi_i^1 = 0$ for $i = 1 + V/k \cdot V$. Also, we will take $\phi_i^2 = \beta$ for $i = 1 + V/k \cdots 2V/k$ and zero for all other i. We will follow this pattern for all other μ, so that for a given i, ϕ_i^μ is non-zero only for one value of μ. The matrix $H_> + \lambda$ has $V - k$ non-zero eigenvalues, and can be thought of as a $(V-k)$-by-$(V-k)$ matrix. Using the particular choice of ϕ_i^μ here, the numerator of the above equation can be written as the product of k determinants, one for each μ. Each determinant is the determinant of a $(V - k - V/k)$-by-$(V - k - V/k)$ random matrix since for the given choice of ϕ, for each μ the factor $\prod_i (\sum_{\mu, \nu} 2\bar{\psi}_a^\mu v_i^a \phi_i^\mu \phi_i^\nu v_i^a \psi_a^\nu)$ removes V/k different factors of $\bar{\psi}^\mu, \psi^\mu$ from the integral. In fact, this result is not specific to this choice of ϕ_i^μ, but is easiest to obtain with this particular choice.

Now, if we evaluate the product of the determinants of the k different $V - k - V/k$-by-$V - k - V/k$ random matrices, and divide by the kth power of the determinant of $H_> + \lambda$, we find that the result is $(2\sqrt{\tau})^V$, times some corrections which change S_k by terms of order $V/\sqrt{k}$. The calculation of the ratio of these determinants is elementary and will not be done here. It is simply a matter of using the Wigner semicircle distribution for the eigenvalues of the different matrices, and evaluating the determinant by taking a product of eigenvalues.

Similarly, there are corrections to the fluctuations in λ_i, involving the contribution to these fluctuations resulting from $(\det(H_> + \lambda_i))^k$. Looking at a perturbative expansion of the determinant similar to that performed above, one would again think that the corrections to the fluctuations change S_k by order V/k, but again the perturbation expansion involves $(G_{ij})^2$ and is invalidated in the spin glass phase. One again finds corrections to S_k of order $V/\sqrt{k}$.

Finally, the corrections to S_k discussed here depend on the average value of λ_i. This can lead to a shift in λ away from the value calculated in Appendix A. These corrections can adjust the value of the gap, $\lambda + a$, but for $k \propto V^{2/5}$ they will not alter the scaling of the gap with k; if we calculate an effective action for λ combining the corrections discussed here with the results from Appendix A, all the terms will be of the same magnitude and $\lambda + a$ will still scale as $(k/V)^{2/3}$.

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